A New Approach to Impulsive Rendezvous near Circular Orbit

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Abstract

A new approach is presented for the problem of optimal impulsive rendezvous of a spacecraft in an inertial frame near a circular orbit in a Newtonian gravitational field. The total characteristic velocity to be minimized is replaced by a related characteristic-value function and this related optimization problem can be solved in closed form. The solution of this problem is shown to approach the solution of the original problem in the limit as the boundary conditions approach those of a circular orbit. Using a form of primer-vector theory the problem is formulated in a way that leads to relatively easy calculation of the optimal velocity increments. A certain vector that can easily be calculated from the boundary conditions determines the number of impulses required for solution of the optimization problem and also is useful in the computation of these velocity increments.

Necessary and sufficient conditions for boundary conditions to require exactly three nonsingular non-degenerate impulses for solution of the related optimal rendezvous problem, and a means of calculating these velocity increments are presented. If necessary these velocity increments could be calculated from a hand calculator containing trigonometric functions. A simple example of a three-impulse rendezvous problem is solved and the resulting trajectory is depicted.

Optimal non-degenerate nonsingular two-impulse rendezvous for the related problem is found to consist of four categories of solutions depending on the four ways the primer vector locus intersects the unit circle. Necessary and sufficient conditions for each category of solutions are presented. The region of the boundary values that admit each category of solutions of the related problem are found, and in each case a closed-form solution of the optimal velocity increments is presented. Some examples are simulated. Similar results are presented for the simpler optimal rendezvous that require only one-impulse. For brevity degenerate and singular solutions are not discussed in detail, but should be presented in a following study

Although this approach is thought to provide simpler computations than existing methods, its main contribution may be in establishing a new approach to the more general problem.

1 Introduction

It has been a convenient practice to model spacecraft orbital and trajectory problems having relatively high thrusts over short intervals of time by discontinuous jumps in velocity, but retaining continuity of the position vector at the time at which the discontinuity in the velocity appears. Problems that are modeled in this way are called **impulsive orbital** or **impulsive trajectory** problems. The impulsive problems considered here are based on the restricted two-body problem, (i.e. a particle of mass in a Newtonian gravitational field emanating from a point source). These minimization problems consist of the determination of a finite set of velocity increments $\Delta\mathbf{v}_1, \dots \Delta\mathbf{v}_k$ and the values of the true anomaly $\theta_1 \dots \theta_k$ at which they are applied, to minimize the total characteristic velocity $\Sigma|\Delta\mathbf{v}_i|$ subject to the two-body equations of motion, the initial position $\mathbf{r}(\theta_0) = \mathbf{r}_0$, initial velocity $\mathbf{v}(\theta_0) = \mathbf{v}_0$, the terminal position $\mathbf{r}(\theta_f) = \mathbf{r}_f$ and the terminal velocity $\mathbf{v}(\theta_f) = \mathbf{v}_f$ where θ_0 and θ_f denote respectively the initial and terminal values of the true anomaly. If $\mathbf{r}_0$, $\mathbf{v}_0$, $\mathbf{r}_f$, $\mathbf{v}_f$ are specified points in $\mathbb{R}^3$ the minimization problem will be called an **optimal rendezvous problem**. If $\mathbf{r}_0$, $\mathbf{v}_0$ or $\mathbf{r}_f$, $\mathbf{v}_f$ or both are arbitrary on a nontrivial arc of a fixed two-body orbit, the minimization problem will be called an **optimal transfer problem**. Clearly, any solution of an optimal transfer problem found at the points $\mathbf{r}(\theta_0), \mathbf{v}(\theta_0), \mathbf{r}(\theta_f), \mathbf{v}(\theta_f)$ also defines a solution of an optimal rendezvous problem having those end conditions.

Apparently the first significant publication of an impulsive orbital problem was the Hohmann Transfer [1], an elliptical arc, tangent to two circular orbits, that appeared in 1925. This was followed in 1929 by the Oberth Transfer [2], a two-impulse transfer connecting a circular orbit to a hyperbolic orbit through an ellipse with apsides touching the circle and the center of attraction. The second impulse at (or near) the center of attraction sends the craft into hyperbolic speed.

Some of the earliest studies of orbital maneuvers were done by Contensou [3, 4] and Lawden [5]. Surveys of much early work were done by Edelbaum [6], Bell [7], Robinson [8], and Gobetz and Doll [9]. An excellent introduction to the subject is found in the book by Marec [10].

In the fifties it was discovered that the total characteristic velocity can be reduced over that of the Hohmann Transfer by the addition of a third impulse if one of the circular orbits is much larger than the other [11, 12, 13]. This might lead one to expect that some minimum total characteristic velocity problems for a single Newtonian gravitational source would require at least three impulses. In 1965 a paper appeared by Marchal [14] showing that a bounded version of the bi-elliptic transfer was indeed a three-impulse optimal transfer.

Recently this problem has been revisited by Pontani [15] who through calculus and a simple graphical technique separated the optimal two-impulse solutions (Hohmann) from the optimal three-impulse solutions (bi-elliptic).

Although three-impulse solutions to the optimal rendezvous problems and the optimal

transfer problems exist, they are somewhat rare in the literature. Ting [16] in 1960 and Marchal [14] in 1965 discovered that at most three impulses are sufficient to solve the optimal rendezvous or transfer problem. There are solutions having more than three impulses, but these solutions are either degenerate or singular solutions, and also have a three-impulse solution containing the same total characteristic velocity as those having more than three-impulses.

Apparently the required number of impulses for optimality is dependent on the way the equations of motion are modeled. In relative-motion studies in which the equations of motion are linearized about a point in Keplerian orbit as many as four impulses can be required for optimality of the planar problem. It has been shown by Neustadt [17], and others [18-20] that the maximum number of impulses needed for linear equations is the same as the number of state variables, that is, four for planar problems. In 1969 Prussing [21] displayed optimal four-impulse rendezvous maneuvers using equations linearized about a point in circular orbit. This problem was revisited much later by Carter and Alvarez [22]. Calculations of several optimal four-impulse rendezvous maneuvers and a degenerate five-impulse rendezvous maneuver were presented by Carter and Brient [20] in 1995 using equations linearized about elliptical orbits.

Optimal velocity increments in impulsive problems are generally calculated in one of three ways: The solution of a Lambert's problem, primer-vector analysis, or parameter optimization such as through a nonlinear programming algorithm. It is common practice to combine these methods in the solution of specific problems. In the present paper we employ a form of primer-vector analysis.

The primer vector was introduced by Lawden [5] as the part of the adjoint vector corresponding to the velocity that satisfied certain necessary conditions for optimality of a trajectory. Lion and Handelsman [23] summarized these necessary conditions and showed how to construct improvement in the velocity increments based on primer vector analysis. Prussing [19] showed that the necessary conditions of Lawden and Lion and Handelsman are also sufficient if the equations of motion are linear.

The present paper uses equations of motion that describe the restricted two-body problem, but the analysis is confined to near-circular orbits. This limitation is caused by an approximation which is highly accurate when the radial speed is small compared with the tangential speed. This simplification has been used successfully by the authors in studies that involve atmospheric drag [24, 25]. We replace the total characteristic velocity which is the original cost function by a related cost function which is useful near a nominal circular orbit, and present closed-form solutions to this related optimization problem. These results should be more accurate than those based on the approximate Clohessy-Wiltshire equations [28] because the linearization used herein is not approximate. We remark that the trajectories described between impulses are not approximations, and the solutions we present are

optimum in terms of the model that we use. We may therefore use the terminology "optimal solutions". These velocity increments, however, are not accurate unless the trajectory is near a circular orbit.

This paper uses a form of primer-vector analysis in a novel way. It has been well known for ages that in the restricted two-body problem, the equation of motion involving the magnitude r of the position vector $\mathbf{r}$ can be transformed to an equation linear in its reciprocal. This forms the basis of a set of linear equations that describe the motion of a particle of mass in the transformed variables. The impulsive minimization problem is then formulated in terms of the resulting linear equations, and is amenable to a specific theory of impulsive linear rendezvous developed by Carter and Brient [26, 20, 27]. The definition of the primer vector for linear equations is in a more general form than that of Lawden[5] or Lion and Handelsman[23], but the more general form is needed for use with the necessary and sufficient conditions for solution of the impulsive minimization problem studied in this paper.

In this new formulation the state variables r, the radial distance from the center of attraction, $\dot{r}$ its time rate of change, and θ , the true anomaly have been replaced by variables y_1, y_2, y_3 to be defined in the next section, called *transformed variables.* The transformed state vector $\mathbf{y} = (y_1, y_2, y_3)^T$ satisfies a linear differential equation. It is straightforward to define a transformed state transition matrix. The primer vectors are shown to be ellipses for this problem. It will be shown that for three-impulse solutions and two-impulse solutions the primer vectors are completely determined by the initial true anomaly θ_0 and the terminal true anomaly θ_f.

Given θ_0 and θ_f the initial transformed state vector $\mathbf{y}_0$ and the terminal transformed state vector $\mathbf{y}_f$ are used to define a generalized boundary point $\mathbf{z}_f(\mathbf{y_0}, \mathbf{y_f})$. The geometric structure of the set of generalized boundary points associated with an optimal primer vector is a convex cone [20]. Given, θ_0 and θ_f these convex cones partition the set of generalized boundary points into a simplex of convex conical sets. The three-dimensional cones contain the generalized boundary points that admit three-impulse solutions. Points on the boundary of such a cone admit degenerate three-impulse solutions, that is, one or more of the three velocity increments is zero. Although three-impulse solutions are somewhat rare in the literature for the restricted two-body problem, this analysis shows that there are plenty of them near circular orbits. It follows also that the two-dimensional cones define areas that admit two-impulse solutions, and the one-dimensional cones, that is, the straight lines admit the one-impulse solutions. Of course the vertex of any cone, that is, the origin, admits only the zero-impulse solution, a coasting trajectory.

For the original problem having boundary conditions in the vicinity of a nominal circular orbit, near optimal velocity increments can be obtained in closed form. For more general boundary conditions the velocity increments calculated from the related problem could be useful as good initial guesses in a numerical optimization program, and should obviate the

need to solve Lambert's problem.

The paper begins with the general formulation of the problem and the related problem for boundary conditions near circular orbits, then presents some analysis and simulations of non-degenerate three-impulse solutions and various kinds of non-degenerate two-impuse solutions of the related problem and precisely determines the sets of boundary conditions corresponding to each type,followed by some results on one-impulse solutions. For brevity detailed discussion of degenerate and singular solutions is postponed to a later work.

2 Building the Model

The equation of motion of a particle in a Newtonian gravitational field about a homogeneous spherical planet is

$$\ddot{r} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.1}$$

where $\mathbf{r}$ is the position vector with respect to the center of the planet, μ is the product of the universal gravitational constant and the mass of the spherical planet, the upper dot indicates differentiation with respect to time t and $r = |\mathbf{r}| = (\mathbf{r} \cdot \mathbf{r})^{1/2}$ where the inside dot denotes the usual inner product, and $|\mathbf{r}|$ represents the Euclidean norm of the vector $\mathbf{r}$.

2.1 The Optimal Impulsive Rendezvous Problem

In polar coordinates (2.1) becomes

$$r\ddot{r} + 2\dot{r}\dot{\theta} = 0 \tag{2.2}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \tag{2.3}$$

where θ represents the true anomaly. The radial and transverse velocities are denoted respectively by $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$. Throughout we shall use subscript notation to refer to any variable x at time t_i as $x_i = x(t_i)$.

We shall consider the addition of velocity increments $\Delta\mathbf{v}_i = (\Delta v_{r_i}, \Delta v_{\theta_i})^T$ at time t_i for $i = 1, ..., k$ such that

$$\lim_{t \to t_i+} \dot{\mathbf{r}}(t) = \lim_{t \to t_i-} \dot{\mathbf{r}}(t) + \Delta\mathbf{v}_i. \tag{2.4}$$

In this formulation r and θ are continuous everywhere and the differential equations (2.2) and (2.3) are satisfied everywhere on an interval $t_0 \leq t \leq t_f$ except at the instants where the velocity satisfies the jump discontinuities (2.4). At the ends of the time interval, (2.4) becomes

$$\lim_{t \to t_0+} \dot{\mathbf{r}}(t) = \dot{\mathbf{r}}(t_0) + \Delta\mathbf{v}_1, \;\; t_1 = t_0 \tag{2.5}$$

$$\dot{\mathbf{r}}(t_f) = \lim_{t \to t_f-} \dot{\mathbf{r}}(t) + \Delta\mathbf{v}_k, \;\; t_k = t_f. \tag{2.6}$$

The rendezvous problem requires that the initial conditions

$$\mathbf{r}(t_0) = \mathbf{r}_0, \;\; \theta(t_0) = \theta_0 \tag{2.7}$$

$$\dot{\mathbf{r}}(t_0) = \mathbf{v}_{r_0}, \;\; \mathbf{v}_\theta(t_0) = \mathbf{v}_{\theta_0} \tag{2.8}$$

and the terminal conditions,

$$\mathbf{r}(t_f) = \mathbf{r}_f, \;\; \theta(t_f) = \theta_f \tag{2.9}$$

$$\dot{\mathbf{r}}(t_f) = \mathbf{v}_{r_f}, \;\; \mathbf{v}_\theta(t_f) = \mathbf{v}_{\theta_f} \tag{2.10}$$

be satisfied. The total characteristics velocity is defined by

$$c = \sum_{i=1}^{k} |\Delta \mathbf{v}_i|. \tag{2.11}$$

The optimal impulsive rendezvous problem is stated as follows:

Find a positive integer k, a finite set $\{t_1, \ldots t_k\}$ on the interval $\{t_0 \leq t \leq t_f\}$, and a set of velocity increments $\{\Delta\mathbf{v}_1, \ldots \Delta\mathbf{v}_k\}$ such that the differential equations (2.2) and (2.3) are satisfied except on the finite set $\{t_1, \ldots t_k\}$ where (2.4)-(2.6) are satisfied, the boundary conditions (2.7)-(2.10) are satisfied, and the total characteristic velocity (2.11) is minimized.

Similarly one can state an optimal impulsive transfer problem by replacing the requirement that the initial conditions, as described by (2.7), (2.8) and the terminal conditions as described by (2.9), (2.10) be fixed with the requirement that either or both be contained on a Keplerian orbit. Mathematically, the optimal impulsive transfer problem is not significantly different from the optimal impulsive rendezvous problem. In fact, an optimal solution satisfying the fixed end conditions (2.7)-(2.10) also defines an optimal solution of the transfer problem from the Keplerian orbit containing the fixed point described by (2.7), (2.8) to the Keplerian orbit that contains the fixed point described by (2.9), (2.10) if the time interval is sufficiently large.

2.2 The Orbit Equation

We sketch the derivation of the orbit equation which has been well known for many years.

Multiplying (2.2) by r, we have the derivative of

$$r^2\dot{\theta} = h. \tag{2.12}$$

The constant h is the angular momentum. We may use (2.12) and the chain rule to change the independent variable in (2.3) to θ. We use primes to indicate differentiation with respect to θ. By this change of variable and noting from (2.12) that θ is monotone in t, the fourth

order system (2.2), (2.3) on the interval $t_0 \leq t \leq t_f$ is replaced by the following third order system on the interval $\theta_0 \leq \theta \leq \theta_f$:

$$r(\theta)r''(\theta) - 2r'(\theta)^2 = r(\theta)^2 - \mu r(\theta)^3/h^2 \tag{2.13}$$

$$h'(\theta) = 0. \tag{2.14}$$

2.3 Transformation to Linear Equations

Employing the well-known change-of-variable $y = 1/r$ in the orbit equation (2.13) and manipulating some symbols, we obtain the linear differential equation:

$$y'' + y = \mu/h^2. \tag{2.15}$$

We now assign the state variables:

$$y_1 = y \tag{2.16}$$

$$y_2 = y' = -\dot{r}/h \tag{2.17}$$

$$y_3 = \mu/h^2. \tag{2.18}$$

In terms of these state variables, the equations (2.13), (2.14) are transformed into the following set of linear equations:

$$y_1' = y_2 \tag{2.19}$$

$$y_2' = -y_1 + y_3 \tag{2.20}$$

$$y_3' = 0. \tag{2.21}$$

The initial conditions are

$$y_1(\theta_0) = y_{10} = \frac{1}{r_0} \tag{2.22}$$

$$y_2(\theta_0) = y_{20} = -\dot{r}(\theta_0)/h_0 = -\frac{v_{r_0}}{r_0 v_{\theta_0}} \tag{2.23}$$

$$y_3(\theta_0) = y_{30} = \frac{\mu}{h_0^2} = \frac{\mu}{(r_0 v_{\theta_0})^2} \tag{2.24}$$

and the terminal conditions are

$$y_1(\theta_f) = y_{1f} = \frac{1}{r_f}. \tag{2.25}$$

$$y_2(\theta_f) = y_{2f} = -\dot{r}(\theta_f)/h_f = -\frac{v_{r_f}}{r_f v_{\theta_f}} \tag{2.26}$$

$$y_3(\theta_f) = y_{3f} = \frac{\mu}{h_f^2} = \frac{\mu}{(r_f v_{\theta_f})^2} \tag{2.27}$$

where the subscript 0 refers to the initial value of a variable and the subscript f refers to the terminal value. Formulas for recovery of the original variables are

$$r = 1/y_1 \tag{2.28}$$

$$\dot{\theta} = y_1^2\sqrt{\mu/y_3} \tag{2.29}$$

$$\dot{r} = -y_2\sqrt{\mu/y_3} \tag{2.30}$$

$$v_\theta = y_1\sqrt{\mu/y_3}. \tag{2.31}$$

2.4 Restatement of the Optimal Impulsive Rendezvous Problem

At $\theta_i,\ i = 1, ..., k$ we increment the velocity $\dot{\mathbf{r}}(\theta_i)$ by $\Delta\mathbf{v_i} = (\Delta v_{r_i}, \Delta v_{\theta_i})^T$ so that (2.4)-(2.6) are satisfied. This causes an instantaneous jump in the state variables y_2 and y_3 but y_1 is continuous. The increments in y_2 and y_3 caused by the increments in the velocity at θ_i are determined respectively from (2.17) and (2.18).

$$\Delta y_{2i} = -\frac{1}{h_i}(\Delta v_{r_i} - \frac{v_{r_i}}{v_{\theta_i}}\Delta v_{\theta_i}),\ \ i = 1, ..., k \tag{2.32}$$

$$\Delta y_{3i} = -\frac{2\mu r_i}{h_i^3}\Delta v_{\theta_i},\ \ i = 1, ..., k. \tag{2.33}$$

We now consider the imposition of a velocity increment $\Delta\mathbf{v}_i = (\Delta v_{r_i}, \Delta v_{\theta_i})^T$ at θ_i for $i = 1, ..., k$ on the interval $\theta_0 \le \theta \le \theta_f$ such that

$$\lim_{\theta\to\theta_i+} y_2(\theta) = \lim_{\theta\to\theta_i-} y_2(\theta) + \Delta y_{2i},\ \ \lim_{\theta\to\theta_i+} y_3(\theta) = \lim_{\theta\to\theta_i-} y_3(\theta) + \Delta y_{3i}. \tag{2.34}$$

If θ_i is an end point (2.34) becomes

$$\lim_{\theta\to\theta_0+} y_2(\theta) = y_2(\theta_0) + \Delta y_{2i},\ \ \lim_{\theta\to\theta_0+} y_3(\theta) = y_3(\theta_0) + \Delta y_{3i} \tag{2.35}$$

$$y_2(\theta_f) = \lim_{\theta\to\theta_f-} y_2(\theta) + \Delta y_{2i},\ \ y_3(\theta_f) = \lim_{\theta\to\theta_f-} y_3(\theta) = \Delta y_{3i}. \tag{2.36}$$

The optimal impulsive rendezvous problem can now be restated as that of finding a positive integer k, a finite set $K = \{\theta_1, ..., \theta_k\}$ on the interval $\theta_0 \le \theta \le \theta_f$, and a set of velocity increments $\{\Delta\mathbf{v}_i \in \mathbb{R}^2, i = 1, .., k\}$ to minimize the total characteristic velocity (2.11) subject to the linear differential equations (2.19)-(2.21) that are valid except on the set K where (2.32)-(2.36) are satisfied, and that satisfy the boundary conditions (2.22)-(2.27).

2.5 Transformed Velocity Increments and the Related Problem

We transform the velocity increments Δv_{r_i} and Δv_{θ_i}, $i = 1, ..., k$ as follows:

$$\Delta V_{1i} = \frac{1}{h_i}\left(\Delta v_{r_i} - \frac{v_{r_i}}{v_{\theta_i}}\Delta v_{\theta_i}\right) \tag{2.37}$$

$$\Delta V_{2i} = \frac{\mu r_i}{h_i^3}\Delta v_{\theta_i}. \tag{2.38}$$

With this transformation, we define the increments $\Delta \mathbf{y}_i \in \mathbb{R}^3$, $i = 1, ..., k$ from (2.32) and (2.33), caused by the velocity increments as

$$\Delta \mathbf{y}_i = B \Delta \mathbf{V}_i \tag{2.39}$$

where $\Delta \mathbf{V}_i = (\Delta V_{1i}, \Delta V_{2i})^T, i = 1, ..., k$ and

$$B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -2 \end{pmatrix}. \tag{2.40}$$

For $i = 1, ..., k$ the expressions (2.34)-(2.36), in vector notation become

$$\lim_{\theta \to \theta_i +} \mathbf{y}(\theta) = \lim_{\theta \to \theta_i -} \mathbf{y}(\theta) + \Delta \mathbf{y}_i, \ \ \theta_0 < \theta_i < \theta_f, \tag{2.41}$$

$$\lim_{\theta \to \theta_0 +} \mathbf{y}(\theta) = \mathbf{y}(\theta_0) + \Delta \mathbf{y}_i, \ \ \theta_i = \theta_0, \tag{2.42}$$

$$\mathbf{y}(\theta_f) = \lim_{\theta \to \theta_0 -} \mathbf{y}(\theta) + \Delta \mathbf{y}_i, \ \ \theta_i = \theta_f. \tag{2.43}$$

In vector form the boundary conditions (2.22)-(2.27) are written

$$\mathbf{y}(\theta_0) = \mathbf{y}_0 \tag{2.44}$$

$$\mathbf{y}(\theta_f) = \mathbf{y}_f \tag{2.45}$$

where $\mathbf{y}_0 = (y_{10}, y_{20}, y_{30})^T$ and $\mathbf{y}_f = (y_{1f}, y_{2f}, y_{3f})^T$ are defined from the respective right-hand sides of (2.22)-(2.24) and (2.25)-(2.27). For $\theta \notin K$ the linear equations (2.19)-(2.21) are

$$\mathbf{y}' = A\mathbf{y} \tag{2.46}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.47}$$

We now replace the cost function (2.11) by the related cost function

$$C = \sum_{i=1}^{k} |\Delta \mathbf{V}_i|. \tag{2.48}$$

The related optimal impulsive rendezvous problem is that of finding a set $K = \{\theta_1, ..., \theta_k\}$ on the interval $\theta_0 \leq \theta \leq \theta_f$ and velocity increments $\Delta\mathbf{V}_1, ..., \Delta\mathbf{V}_k \in \mathbb{R}^2$ that minimize the related cost function C subject to the linear differential equation (2.46) that is satisfied everywhere on the interval except the set K where the velocity increments $\Delta\mathbf{V}_1, ..., \Delta\mathbf{V}_k$ are applied subject to (2.39) and (2.41)-(2.43), and the initial condition (2.44) and the terminal condition (2.45) are satisfied.

It has been shown [20] that a solution of this problem exists under appropriate controllability conditions.

Since the differential equation (2.46) is linear, it is known [17-20] that it is sufficient to set $k = 3$. For this reason we set $k = 3$ for the remainder of this paper. This agrees with the results of Ting [16] and Marchal [14] although the linearized model used by Prussing [21] allows $k = 4$.

2.6 Necessary and Sufficient Conditions

A fundamental matrix solution Φ associated with a matrix A satisfies the matricial differential equation $\Phi' = A\Phi$. It is not difficult to see that

$$\Phi(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 1 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.49}$$

is a fundamental matrix solution associated with (2.47). The inverse of Φ is

$$\Phi^{-1}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & -\cos\theta \\ \sin\theta & \cos\theta & -\sin\theta \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.50}$$

Lawden's definition [5] of the primer vector is not adequate for the work herein. We use the definition [20,26]:

$$\mathbf{p}(\theta) = R(\theta)^T \lambda \tag{2.51}$$

where $\lambda \in \mathbb{R}^3$ and

$$R(\theta) = \Phi^{-1}(\theta) B. \tag{2.52}$$

We briefly review some previous results [20].

If $\Delta\mathbf{V}_i$ is a velocity impulse at θ_i it follows from (2.39), (2.46), and (2.41)-(2.43) that

$$\mathbf{y}(\theta) = \Phi(\theta)\Phi(\theta_i)^{-1}(\mathbf{y}_i + B\Delta\mathbf{V}_i) \tag{2.53}$$

on the interval $\theta_i < \theta < \theta_{i+1}$ if there is a velocity impulse at θ_{i+1}, otherwise $\theta_i < \theta < \theta_f$. If there are a total of k impulses it follows from (2.39), (2.41)-(2.44) and (2.53) that

$$\mathbf{y}(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}\mathbf{y}_0 + \Phi(\theta)\sum_{i=1}^{k}\Phi(\theta_i)^{-1}B\Delta\mathbf{V}_i \tag{2.54}$$

where $\theta_k \leq \theta \leq \theta_f$. Applying the terminal condition (2.45) we obtain

$$\mathbf{y}_f = \Phi(\theta_f)\Phi(\theta_0)^{-1}\mathbf{y}_0 + \Phi(\theta_f)\sum_{i=1}^{k}\Phi(\theta_i)^{-1}B\Delta\mathbf{V}_i. \tag{2.55}$$

Introducing the vector $\mathbf{z}_f$, we write this expression as

$$\mathbf{z}_f = \sum_{i=1}^{k}\Phi(\theta_i)^{-1}B\Delta\mathbf{V}_i \tag{2.56}$$

where all of the information about the boundary conditions is contained in the definition:

$$\mathbf{z}_f = \Phi(\theta_f)^{-1}\mathbf{y}_f - \Phi(\theta_0)^{-1}\mathbf{y}_0. \tag{2.57}$$

Setting $\alpha_i = |\Delta\mathbf{V_i}|$, $i = 1 \ldots k$ we have all of the terminology needed to state the following result from previous work.[20]

Theorem: For a minimizing k-impulse solution of the related optimal impulsive rendezvous problem, it is necessary and sufficient that

$$\Delta\mathbf{V}_i = 0, \;\; or \;\; \Delta\mathbf{V}_i = -p(\theta_i)\alpha_i, \;\; i = 1, \ldots, k \tag{2.58}$$

$$\alpha_i \geq 0, \;\; i = 1, \ldots, k \tag{2.59}$$

$$\sum_{i=1}^{k} R(\theta_i)p(\theta_i)\alpha_i = -z_f \tag{2.60}$$

$$\Delta\mathbf{V}_i = 0, \;\; or \;\; |p(\theta_i)| = 1, \;\; i = 1, \ldots, k \tag{2.61}$$

$$\theta_i = \theta_0, \;\; or \;\; |p(\theta_i)|' = 0, \;\; or \;\; \theta_i = \theta_f, \;\; i = 1, \ldots, k \tag{2.62}$$

$$|p(\theta)| \leq 1, \;\; \theta_0 \leq \theta \leq \theta_f. \tag{2.63}$$

Proof: See Sec 3.2 and 3.3 of Reference 20.

In this theorem k is the specified number of impulses and can be any non-negative integer sufficiently large that (2.58)-(2.63) are satisfied. It has been shown [17-20] to be unnecessary

for k to be larger than the dimension of the state space (i.e. dim $\mathbf{z}_f$ which is 3 for the present problem), although there can be degenerate solutions where $k > dim\, \mathbf{z}_f$. A **degenerate solution** is a minimizing solution for which there is an equivalent minimizing solution where some $\alpha_i = 0$ in (2.59). If a solution is degenerate an equivalent minimizing solution exists having fewer than k impulses so that, effectively, the number k could be reduced.

For many boundary conditions there can be minimizing solutions for $k < dim\, \mathbf{z}_f$. For this reason we shall set k equal to the number of elements in the set

$$K = \{\theta|\, |\mathbf{p}(\theta)| = 0, \;\; \theta_0 \leq \theta \leq \theta_f\}$$

if this set is finite. Since the function $f(\theta) = \mathbf{p}^T(\theta)\mathbf{p}(\theta) - 1$ is analytic either K is finite (i.e. $K = \{\theta_1, \ldots, \theta_k\}$) or else K is the entire interval $\theta_0 \leq \theta \leq \theta_f$. In the latter case a minimizing solution is called **singular** and k and $\theta_1, \ldots, \theta_k$ are arbitrary as long as k is large enough that (2.58)-(2.63) are satisfied, as they will be if $k = dim\, \mathbf{z}_f$.

In the present paper we shall isolate those boundary conditions where $k = 3$ and investigate minimizing non-degenerate three-impulse solutions. In the following paper we shall investigate the non-degenerate two-impulse and one-impulse solutions. In the final paper we shall present some results on singular and degenerate solutions.

We shall introduce some terminology based on (2.62). We refer to θ_0 as an *initial value* of θ and θ_f as a *terminal value* of θ. If an optimal impulsive rendezvous has a velocity increment assigned at θ_0 (i.e. $\Delta\mathbf{V}_i = \Delta\mathbf{V}_0$) it is called an *initial velocity impulse*; if it has a velocity increment assigned at θ_f (i.e. $\Delta\mathbf{V}_i = \Delta\mathbf{V}_f$) it is called a *terminal velocity impulse.* A value θ_i is called a *stationary value* if $|p(\theta_i)|' = 0$ and $p(\theta_i) = 1$. A velocity increment $\Delta\mathbf{V}_i$ assigned at a stationary value θ_i is called *a stationary velocity impulse.* If the one-sided derivative of $|p(\theta)|$ is zero at θ_0, then θ_0 is both an *initial value* and a *stationary value*, and $\Delta\mathbf{V}_0$ is both an *initial* and a *stationary velocity impulse.* Similarly, if the one-sided derivative of $|p(\theta)|$ is zero at θ_f, then θ_f is both a *terminal value* and a *stationary value* and $\Delta\mathbf{V}_f$ is both a terminal and a stationary velocity increment.

2.7 Optimal Impulsive Rendezvous near a Nominal Orbit of Low Eccentricity

Given θ_0, $\mathbf{y}_0$, θ_f, $\mathbf{y}_f$ an analytical solution will be presented for the related optimal impulsive rendezvous problem based on the necessary and sufficient conditions recently stated. It is possible to display the velocity increments, their points of application, and the actual arcs of Keplerian orbits that minimize the related cost function subject to (2.39)-(2.47). A practical limitation of this result is that it is the total characteristic velocity (2.11) that we seek to minimize, not the related cost function (2.48).

We shall show that the total characteristic velocity is approximately proportional to the related cost function and approaches exactness as $\mathbf{y}_0$ and $\mathbf{y}_f$ approach end conditions associated with a circular orbit.

Consider a state vector $\bar{\mathbf{y}}(\theta)$ associated with a nominal Keplerian orbit where $\bar{\mathbf{y}}(\theta_0) = \bar{\mathbf{y}}_0$ and $\bar{\mathbf{y}}(\theta_f) = \bar{\mathbf{y}}_f$. Since there are no impulses in the nominal orbit, (2.55) shows that

$$\bar{\mathbf{y}}_f = \Phi(\theta_f)\Phi(\theta_0)^{-1}\bar{\mathbf{y}}_0. \tag{2.64}$$

Comparing an impulsive orbit with a nominal orbit,

$$\mathbf{y}(\theta) - \bar{\mathbf{y}}(\theta) = \Phi(\theta)\Phi(\theta_0)^{-1}(\mathbf{y}_0 - \bar{\mathbf{y}}_0) + \Phi(\theta)\sum_{i=1}^{k}\Phi(\theta_i)^{-1}B\Delta\mathbf{V}_i \tag{2.65}$$

where $k = 1, 2$, or 3. From (2.56) and (2.57) we observe that

$$\mathbf{z}_f - \bar{\mathbf{z}}_f = \Phi(\theta_f)^{-1}(\mathbf{y}_f - \bar{\mathbf{y}}_f) - \Phi(\theta_0)^{-1}(\mathbf{y}_0 - \bar{\mathbf{y}}_0) \tag{2.66}$$

where $\bar{\mathbf{z}}_f = 0$ from (2.56). This shows that $\mathbf{z}_f$ can be made arbitrarily small by making $\mathbf{y}_f$ sufficiently near $\bar{\mathbf{y}}_f$ and $\mathbf{y}_0$ sufficiently near $\bar{\mathbf{y}}_0$. Upon multiplying both sides of (2.56) by a very small positive number, we see that we can select $\epsilon > 0$ and $|\Delta\mathbf{V}_i| < \epsilon/3$, $i = 1, 2, 3$ where ϵ is made arbitrarily small by selecting $\mathbf{z}_f$ sufficiently small. In this manner we can make the cost $C < \epsilon$ and the difference (2.65) in the trajectories arbitrary small.

We shall choose the nominal trajectory $\bar{\mathbf{y}}(\theta)$ to be a circular orbit or a low-eccentricity elliptical orbit. Picking end conditions sufficiently near the end conditions of the nominal orbit we establish a sufficiently small bound on $\Delta\mathbf{V}_i$, $i = 1, 2, 3$ (i.e. there is a sufficiently small positive number M such that $|\Delta\mathbf{V}_i| < M$ for $i = 1, 2, 3$). This establishes a bound N on $|\Delta v_{\theta_i}|$, for $i = 1, 2, 3$ (i.e. there is a number $N > 0$ such that $|\Delta v_{\theta_i}| < N$ for $i = 1, 2, 3$). We have shown that the difference (2.65) from the nominal circular or low-eccentricity orbit can be made as small as possible. This and (2.17) show that $|v_{r_i}|$ is much less than $|v_{\theta_i}|$ near a low-eccentricity nominal orbit. The conclusion of these arguments is that the second term in the parenthesis on the right-hand side of (2.37) can be made arbitrarily small by picking the end conditions sufficiently near those of the nominal circular orbit.

One way to select a radius defining a nominal circular orbit is by setting $\bar{r} = (r_0 + r_f)/2$. It is not difficult to show that the angular momentum $\bar{h}$ of a circular orbit of radius $\bar{r}$ is

$$\bar{h} = (\mu\bar{r})^{1/2}. \tag{2.67}$$

Using $k = 3$ we now show that the total characteristic velocity (2.11) can be approximated to any desired accuracy using the related cost (2.48) by picking the end conditions sufficiently near these of a nominal circular orbit.

It follows from (2.37) that for $i = 1, 2, 3$

$$|\Delta \mathbf{v}_i - h_i \Delta \mathbf{V}_i| = \left[(\frac{v_{r_i}}{v_{\theta_i}})^2 + (1 - \frac{\mu r_i}{h_i^2})^2 \right]^{1/2} |\Delta v_{\theta_i}|. \quad (2.68)$$

Given any $\epsilon > 0$, , there are end points sufficiently near end points of a nominal circular orbit such that for $i = 1, 2, 3$

$$\left| \frac{v_{r_i}}{v_{\theta_i}} \right| < \frac{\epsilon}{6N}, \quad \left| \frac{\mu \bar{r}}{\bar{h}^2} - \frac{\mu r_i}{h_i^2} \right| < \frac{\epsilon}{6N}. \quad (2.69)$$

It follows from (2.67) and the fact that $|\Delta v_{\theta_i}| < N$ for $i = 1, 2, 3$ that

$$|\Delta \mathbf{v}_i - h_i \Delta \mathbf{V}_i| < \frac{\epsilon}{6}. \quad (2.70)$$

To complete the arguments, we note that

$$|\sum_{i=1}^{3} |\Delta \mathbf{v}_i| - \bar{h} \sum_{i=1}^{3} |\Delta \mathbf{V}_i|| \leq \sum_{i=1}^{3} |\Delta \mathbf{v}_i - \bar{h} \Delta \mathbf{V}_i| \leq \sum_{i=1}^{3} |\Delta \mathbf{v}_i - h_i \Delta \mathbf{V}_i| + \sum_{i=1}^{3} |h_i - \bar{h}||\Delta \mathbf{V}_i|. \quad (2.71)$$

Picking initial and final conditions sufficiently near the nominal circular orbit, for each $i = 1, 2, 3$ we have

$$|h_i - \bar{h}| < \frac{\epsilon}{6M}. \quad (2.72)$$

This establishes the final result:

$$|\sum_{i=1}^{3} |\Delta \mathbf{v}_i| - \bar{h} \sum_{i=1}^{3} |\Delta \mathbf{V}_i|| < \epsilon \quad (2.73)$$

For trajectories near a nominal orbit of low eccentricity we have the approximation

$$\sum_{i=1}^{3} |\Delta \mathbf{v}_i| \cong \bar{h} \sum_{i=1}^{3} |\Delta \mathbf{V}_i|. \quad (2.74)$$

Since $\bar{h}$ is a constant we may use the cost

$$C = \sum_{i=1}^{3} |\Delta \mathbf{V}_i| \quad (2.75)$$

in these situations. After solution of this related problem one can calculate the actual impulses $\Delta \mathbf{v}_i$ in terms of $\Delta \mathbf{V}_i$ through (2.37) and (2.38) where v_{r_i} and v_{θ_i} are recovered from (2.28) - (2.31). If the end conditions are very close to a nominal circular orbit one can simplify and approximate (2.37) and (2.38) by

$$\Delta \mathbf{v}_i = \bar{h} \Delta \mathbf{V}_i, \ \ i = 1, 2, 3. \quad (2.76)$$

3 Primer Vector Analysis

The primer vector can be a very useful tool [5, 23, 19, 20] in the analysis of optimal impulsive problems. In this section we survey the various geometric arrangements of primer vector loci for the problem addressed, its degeneration in the case of singular solutions, and its use in the calculation of three-impulse trajectories.

3.1 Geometry of Primer Vector Loci

From (2.40), (2.50), (2.52) and (2.51) the primer vector is determined in terms of the vector λ. Writing $\lambda^T = (\lambda_1, \lambda_2, \lambda_3)$ and $\mathbf{p}(\theta)^T = (p_1(\theta), p_2(\theta))$ it follows that

$$p_1(\theta) = \lambda_1 \sin\theta - \lambda_2 \cos\theta \tag{3.1}$$

$$p_2(\theta) = 2\lambda_1 \cos\theta + 2\lambda_2 \sin\theta - 2\lambda_3 \tag{3.2}$$

We set $\lambda_1 = \lambda \cos\phi$ and $\lambda_2 = \lambda \sin\phi$ where $\lambda = (\lambda_1^2 + \lambda_2^2)^{1/2}$. The primer vector is therefore described by

$$p_1(\theta) = \lambda \sin(\theta - \phi) \tag{3.3}$$

$$p_2(\theta) = 2\lambda \cos(\theta - \phi) - 2\lambda_3 \tag{3.4}$$

If $\lambda \neq 0$ then the primer vector loci are ellipses. The equations (3.3) and (3.4) can be combined and put in the standard form:

$$\frac{p_1^2}{\lambda^2} + \frac{(p_2 + 2\lambda_3)^2}{(2\lambda)^2} = 1. \tag{3.5}$$

The major axis is 2λ, the minor axis is λ, and the center is at $(-2\lambda_3, 0)$ where the p_2-axis is the ordinate and the p_1-axis is the abscissa. According to (2.61) it is necessary that optimal velocity impulses are applied at the values of θ_i described by the intersections of the ellipse (3.5) and the unit circle. Also (2.63) shows that the primer vector arc must be contained inside the unit circle. This is depicted graphically in Figures 1-8 for various types of three-impulse, two-impulse, and one-impulse solutions. The arrows indicate the points of application of the velocity increments.

If θ is any value of the true anomaly we let $\bar{\theta} = \theta - \phi$. Similarly for a subscript i we let $\bar{\theta}_i = \theta_i - \phi$. We therefore write (3.3) and (3.4) as

$$p_1(\theta) = \lambda \sin\bar{\theta} \tag{3.6}$$

$$p_2(\theta) = 2\lambda(\cos\bar{\theta} - \kappa) \tag{3.7}$$

where

$$\kappa = \frac{\lambda_3}{\lambda}. \tag{3.8}$$

If (3.6) and (3.7) are associated with an optimal solution for a generalized boundary point $\mathbf{z}_f$ and optimal velocity increments are $\Delta\mathbf{V_i}$, $i = 1, ..., k$, we see that

$$p_1(\theta) = -\lambda \sin \bar{\theta} \tag{3.9}$$

$$p_2(\theta) = -2\lambda(\cos \bar{\theta} - \kappa) \tag{3.10}$$

are associated with the generalized boundary point $-\mathbf{z}_f$ and optimal velocity increments are $-\Delta\mathbf{V_i}$, $i = 1, ..., k$. This latter solution for the generalized boundary point $-\mathbf{z}_f$ will be called the **antipodal** solution relative to the original solution associated with $\mathbf{z}_f$. Obviously if the original solution has been determined, the antipodal solution is available without calculation. Figure 2 depicts the locus of a primer vector that is antipodal to that presented in Figure 1. This concept is not found much in the literature although Breakwell does refer to "mirror image" trajectories [29].

3.2 Singular Solutions

An arc of a trajectory is called *singular* if $|p(\theta)| = 1$ identically on an interval. If $\lambda = 0$, then (3.3), (3.4),and (2.61) show that $\lambda_3 = \pm 1/2$ and

$$p_1(\theta) = 0 \tag{3.11}$$

$$p_2(\theta) = \pm 1 \tag{3.12}$$

everywhere on the interval $\theta_0 \leq \theta \leq \theta_f$. There are no other values of λ_1, λ_2, or λ_3 for which singular solutions occur. The two constant primer vectors described by (3.11) and (3.12) will define two optimal singular solutions on $\theta_0 \leq \theta \leq \theta_f$. From (2.58)-(2.60) we find that

$$\Delta\mathbf{V}_i = \pm \begin{pmatrix} 0 \\ \alpha_i \end{pmatrix}, \quad i = 1, ..., k \tag{3.13}$$

$$\sum_{i=1}^{k} \alpha_i \cos \theta_i = \pm z_{f_1}/2 \tag{3.14}$$

$$\sum_{i=1}^{k} \alpha_i \sin \theta_i = \mp z_{f_2}/2 \tag{3.15}$$

$$\sum_{i=1}^{k} \alpha_i = \pm z_{f_3}/2 \tag{3.16}$$

where k is an arbitrary positive integer. We note that terms determined from the lower algebraic signs represent antipodal solutions. We also observe that although the optimal velocity increments must satisfy the form (3.13), and the sums on the left-hand sides must

match the specified end conditions on the right-hand sides, these velocity increments are otherwise arbitrary. In these singular cases there are non-unique optimal trajectories that satisfy the end conditions and minimize the related cost function (2.48). We note that the left-hand side of (3.16) is the minimum cost (2.48). We note that for the singular solutions all velocity increments are stationary velocity impulses. In terms of the original variables r and θ, the singular solutions describe an outward (or inward) multi-impulse spiral from (θ_0, r_0) to (θ_f, r_f).

3.3 Three-Impulse Solutions

For the remainder of this first paper we set $k = 3$ in (2.58)-(2.63).

According to (2.61), for three-impulse solutions it is necessary that optimal velocity increments be applied where $|\mathbf{p}(\theta_i)| = 1,\ i = 1, 2, 3$. For primer vector equations (3.6) and (3.7) this becomes

$$\lambda^2[\sin^2 \bar{\theta}_i + 4(\cos \bar{\theta}_i - \kappa)^2] = 1 \tag{3.17}$$

for $i = 1, 2, 3$.

3.3.1 Stationary Solutions

We first consider three-impulse solutions in which all the velocity increments are stationary velocity increments. These occur at values θ_i where (2.62) and (3.17) are satisfied. Using (3.6) and (3.7) with (2.62) we find that if θ_i is a stationary value then it is necessary that

$$\sin \bar{\theta}_i \cos \bar{\theta}_i - 4 \sin \bar{\theta}_i(\cos \bar{\theta}_i - \kappa) = 0. \tag{3.18}$$

This implies that either 1) $\sin \bar{\theta}_i = 0$ or 2) $\kappa = 3/4 \cos \bar{\theta}_i$.

We show now that the second alternative is not feasible because it violates (2.63). Using (3.6) and (3.7) and forming the function

$$f(\theta) = p_1(\theta)^2 + p_2(\theta)^2 - 1 \tag{3.19}$$

we find that if $f'(\theta_i) = 0$ where $\bar{\theta}_i$ satisfies 2) then $f''(\theta_i) > 0$ if $\cos^2(\bar{\theta}_i) \neq 1$. It so happens that $\cos^2(\bar{\theta}_i) = 1$ implies that the first alternative 1) is satisfied. Our argument shows that $|\mathbf{p}(\theta)| \geq 1$ in a neighborhood of θ_i and equality holds in that neighborhood only at θ_i if the second alternative 2) is satisfied.

We therefore conclude that the first alternative 1) must be satisfied and

$$\bar{\theta}_i = \pi i, \quad i = 0, 1, 2, \ldots . \tag{3.20}$$

Using two successive values of (3.20) in (3.17) we obtain

$$2\lambda(1 - \kappa) = 1 \tag{3.21}$$

$$2\lambda(1+\kappa) = 1. \tag{3.22}$$

Clearly $\lambda = 1/2$ and $\kappa = 0$ so the primer vector equations (3.6) and (3.7) respectively become

$$p_1(\theta) = \frac{1}{2}\sin\bar{\theta} \tag{3.23}$$

$$p_2(\theta) = \cos\bar{\theta}. \tag{3.24}$$

Forming the function (3.19) we see that $|\mathbf{p}(\theta)| < 1$ except at θ_i satisfying (3.20) where equality holds, as seen in Figure 3.

We shall pursue this problem further in our later examination of multi-impulse and degenerate solutions. It suffices to state at this point that if we substitute (3.23) and (3.24) into (2.60), there are not unique solutions for $\alpha_1, ..., \alpha_k$ for $k > 2$. The three-impulse stationary solutions are therefore degenerate, and can be replaced by two-impulse solutions.

In seeking non-degenerate three-impulse solutions, we must consider solutions in which all of the velocity increments are not applied at stationary values.

3.3.2 Non-Degenerate Three-Impulse Solutions

Throughout this paper we adopt the convention that the motion of increasing θ is counter clockwise and that $\theta_0 < \theta_f$. If $-\pi \leq \bar{\theta}_0 \leq \pi$ there are two cases, $-\pi < \bar{\theta}_0 < 0$ and $0 < \bar{\theta}_0 < \pi$. It is well known that there are at most four intersections of an ellipse such as (3.5) with the unit circle. For the first case the intersections can be represented using at most four values $\bar{\theta}_i,\ i = 1,2,3,4$ where $-\pi \leq \bar{\theta}_i \leq \pi,\ i = 1,2,3,4$. For the second case the four values representing intersections are considered on the interval $0 \leq \bar{\theta}_i \leq 2\pi,\ i = 1,2,3,4$.

We present the following lemma establishing necessary conditions for optimal non-degenerate three-impulse solutions. The primer vector loci for these solutions are depicted in Figure 1 or Figure 2. The primer vector is defined by (3.6)-(3.8) and the function f is defined by (3.19).

Lemma: It is necessary that an optimal non-degenerate three-impulse solution satisfy the following:

$$\theta_1 = \theta_0, \ \ \theta_3 = \theta_f \tag{3.25}$$

$$\theta_2 = \frac{1}{2}(\theta_0 + \theta_f) \tag{3.26}$$

$$0 < \theta_f - \theta_0 < 2\pi \tag{3.27}$$

$$f'(\theta_0) = -f'(\theta_f) < 0 \tag{3.28}$$

$$f(\theta) \leq 0, \quad \theta_0 \leq \theta \leq \theta_f. \tag{3.29}$$

There is an integer i such that either $2i\pi < \bar{\theta}_0 < (2i+1)\pi$ and

$$\bar{\theta}_2 = (2i+1)\pi,\ \mathit{and}\ \phi = \theta_2 - (2i+1)\pi \tag{3.30}$$

$$\kappa = -\frac{3}{8}(1 - \cos\bar\theta_0), \quad -\frac{3}{4} < \kappa < 0 \tag{3.31}$$

$$\lambda = \frac{4}{5 + 3\cos\bar\theta_0}, \quad \frac{1}{2} < \lambda < 2 \tag{3.32}$$

or $(2i-1)\pi < \bar\theta_0 < 2i\pi$ and

$$\bar\theta_2 = 2i\pi, \ \textit{and}\ \phi = \theta_2 - 2i\pi \tag{3.33}$$

$$\kappa = \frac{3}{8}(1 + \cos\bar\theta_0), \quad 0 < \kappa < \frac{3}{4} \tag{3.34}$$

$$\lambda = \frac{4}{5 - 3\cos\bar\theta_0}, \quad \frac{1}{2} < \lambda < 2. \tag{3.35}$$

Proof: If a solution is non degenerate there must be a value $\bar\theta_m$ on $\bar\theta_0 \le \bar\theta \le \bar\theta_f$ where $\bar\theta_m \ne \pi i$ for any integer i and $\bar\theta_m$ satisfies (3.17), otherwise all velocity increments are at stationary values and the solution is degenerate. For any integer j the value $\bar\theta_n = 2\pi j - \bar\theta_m$ is distinct from $\bar\theta_m$ and satisfies (3.17) also. We let $\bar\theta_1$ be the smaller of the numbers $\bar\theta_m$ and $\bar\theta_n$ and $\bar\theta_3$ be the larger, and $\frac{1}{2}(\bar\theta_1 + \bar\theta_3) = \pi j$.

There must also be a third velocity increment on a value $\bar\theta_2$ satisfying (3.17) where $\bar\theta_0 < \bar\theta_2 < \bar\theta_f$. For any integer l the number $2\pi l - \bar\theta_2$ also satisfies (3.17). Since there can be only three impulses, this number is distinct from $\bar\theta_1$ and $\bar\theta_3$, and we must have $2\pi l - \bar\theta_2 = \bar\theta_2$, consequently $\bar\theta_2 = \pi l$ for some integer l.

We consider first the case where l is odd. We therefore write $l = 2i + 1$ for some integer i. Substituting $\bar\theta_1$ and $\bar\theta_2$ into (3.17) and equating the left-hand sides, we obtain

$$\sin^2\bar\theta_1 + 4(\cos\bar\theta_1 - \kappa)^2 = 4(1+\kappa)^2.$$

Expanding and simplifying we find that

$$3(1 - \cos^2\bar\theta_1) + 8\kappa(1 + \cos\bar\theta_1) = 0.$$

This says that either $\cos\bar\theta_1 = -1$ or

$$\kappa = -\frac{3}{8}(1 - \cos\bar\theta_1).$$

Since $\bar\theta_1 \ne \pi i$ for any integer i, we have the latter. Differentiating $f(\theta)$, evaluating the derivative at θ_1, and substituting the above formula for κ, we obtain

$$f'(\theta_1) = \mathbf{p}(\theta_1)\cdot\mathbf{p}'(\theta_1) = -\frac{3}{2}\lambda^2\sin\bar\theta_1(1 + \cos\bar\theta_1).$$

Next we observe that since $\bar\theta_1 + \bar\theta_3 = 2\pi j$ then $\theta_3 = 2\pi j - \bar\theta_1 + \phi$ and

$$f'(\theta_3) = \frac{3}{2}\lambda^2\sin\bar\theta_1(1 + \cos\bar\theta_1).$$

If $\sin\bar{\theta}_1 < 0$ then $f'(\theta_1) > 0$ and $f'(\theta_3) < 0$. This says that the primer vector exits the unit disk at $\bar{\theta}_1$ and enters at $\bar{\theta}_3$ thus $\bar{\theta}_1$, $\bar{\theta}_2$ and $\bar{\theta}_3$ cannot support a three-impulse solution since $\bar{\theta}_1 < \bar{\theta}_3$ and (2.63) must be satisfied.

As a consequence of this argument, it is necessary that $\sin\bar{\theta}_1 > 0$ hence $f'(\theta_1) < 0$ and $f'(\theta_3) > 0$. In order for a three-impulse solution to be supported on $\bar{\theta}_1$, $\bar{\theta}_2$ and $\bar{\theta}_3$ and satisfy (2.63) it is necessary that $\bar{\theta}_1 = \bar{\theta}_0$ and $\bar{\theta}_3 = \bar{\theta}_f$. It follows that $\sin\bar{\theta}_0 > 0$ resulting in the restriction $2i\pi < \bar{\theta}_0 < (2i+1)\pi < \bar{\theta}_f < (2i+2)\pi$. (Recall that $l = 2i+1$). The only integers that satisfy the restriction are $l = j = 1$, establishing (3.25)-(3.31) for the first case recalling the definition of $\bar{\theta}_i$. Substituting (3.31) into (3.17) and solving for λ produces (3.32).

We consider next the case where l is even. We therefore write $l = 2i$ for some integer i. Substituting $\bar{\theta}_1$ and $\bar{\theta}_2$ into (3.17) and equating the left-hand sides again,

$$\sin^2\bar{\theta}_1 + 4(\cos\bar{\theta}_1 - \kappa)^2 = 4(1-\kappa)^2.$$

Expanding and simplifying as before we find

$$\kappa = \frac{3}{8}(1+\cos\bar{\theta}_1).$$

Substituting this value of κ into $f'(\theta_1)$ we get

$$f'(\theta_1) = \mathbf{p}(\theta_1)\cdot\mathbf{p}'(\theta_1) = \frac{3}{2}\lambda^2\sin\bar{\theta}_1(1-\cos\bar{\theta}_1)$$

and similarly

$$f'(\theta_3)) = -\frac{3}{2}\lambda^2\sin\bar{\theta}_1(1-\cos\bar{\theta}_1).$$

If $\sin\bar{\theta}_1 > 0$ then $f'(\theta_1) > 0$, $f'(\theta_3) < 0$ and we again find that a three-impulse solution cannot be supported on $\bar{\theta}_1$,$\bar{\theta}_2$ and $\bar{\theta}_3$ and satisfy (2.63) because $\bar{\theta}_1 < \bar{\theta}_3$.

It is therefore necessary that $\sin\bar{\theta}_1 < 0$, hence $f'(\theta_1) < 0$ and $f'(\theta_3) > 0$. In order that a three-impulse solution be supported on $\bar{\theta}_1$,$\bar{\theta}_2$ and $\bar{\theta}_3$ and satisfy (2.63), it is again necessary that $\bar{\theta}_1 = \bar{\theta}_0$ and $\bar{\theta}_3 = \bar{\theta}_f$. This requires that $\sin\bar{\theta}_0 < 0$ resulting in the restriction $(2i-1)\pi < \bar{\theta}_0 < 2i\pi < \bar{\theta}_f < (2i+1)\pi$. (Recall that $l = 2i$). The only integers satisfying this restriction are $l = j = 0$ establishing (3.25)-(3.29), recalling again the definition of $\bar{\theta}_i$; (3.33) and (3.34) follow as well. Substituting (3.34) into (3.17) and solving for λ reveals the formula (3.35). ■

Some of the statements in this theorem can be expressed differently by adding and subtracting $\bar{\theta}_2$ from $\bar{\theta}_0$ in (3.31), (3.32), (3.34) and (3.35). We restate the lemma as the following:

Corollary: It is necessary that a non-degenerate three-impulse solution satisfy (3.25)-(3.29), and

$$\lambda = \frac{4}{5 - 3\cos(\frac{\theta_f - \theta_0}{2})}. \tag{3.36}$$

Either

$$\kappa = -\frac{3}{8}\left[1 + \cos(\frac{\theta_f - \theta_0}{2})\right] \tag{3.37}$$

and the primer vector satisfies (3.9) and (3.10) or

$$\kappa = \frac{3}{8}\left[1 + \cos(\frac{\theta_f - \theta_0}{2})\right] \tag{3.38}$$

and the primer satisfies (3.6) and (3.7).

An immediate consequence of the lemma and the necessary and sufficient conditions (2.58)-(2.63) is the following fundamental theorem of non-degenerate three-impulse solutions.

Theorem: The optimal impulsive rendezvous problem defined by (2.40) and (2.47) on the interval $\theta_0 \leq \theta \leq \theta_f$ has a non-degenerate three-impulse solution $\{\theta_1, \theta_2, \theta_3, \Delta\mathbf{V}_1, \Delta\mathbf{V}_2, \Delta\mathbf{V}_3\}$ if and only if (3.25)-(3.30) and (3.36) are satisfied, and either (3.9), (3.10) and (3.37) are satisfied or (3.6), (3.7) (3.33) and (3.38) are satisfied, and

$$\Delta\mathbf{V}_i = -\alpha_i \mathbf{p}(\theta_i), \quad i = 1, 2, 3 \tag{3.39}$$

$$\sum_{1=1}^{3} R(\theta_i)\mathbf{p}(\theta_i)\alpha_i = -\mathbf{z}_f, \tag{3.40}$$

$$\alpha_i > 0, \;\; i = 1, 2, 3, \tag{3.41}$$

where $R(\theta)$ is given from (2.52) and $\mathbf{z}_f$ from (2.57).

From (3.31), (3.32), (3.34) and (3.35)) we see the bounds on λ and κ for which three-impulse solutions exist. These boundary primer vector loci are illustrated in Figure 3.

We remark that an optimal solution in which (3.30)-(3.32) are satisfied has an associated antipodal solution satisfying (3.33)-(3.35) instead. Similarly if a generalized boundary point $\mathbf{z}_f$ has κ satisfying (3.37) then its antipodal solution for the point$-\mathbf{z}_f$ has κ satisfying (3.38).

3.3.3 Finding Non-Degenerate Three-Impulse Solutions

We now apply the preceding theorem to find non-degenerate three impulse solutions.

We consider the case where $\phi = \theta_2 - 2i\pi$ and κ and λ are given respectively by (3.34) and (3.35), θ_0 and θ_f are known and θ_2 is known from (3.26). Setting $\hat{\theta} = \theta - \theta_2$ the primer vector from (3.6) and (3.7) becomes

$$p_1(\theta) = \lambda \sin\hat{\theta} \tag{3.42}$$

$$p_2(\theta) = 2\lambda(\cos\hat{\theta} - \kappa) \tag{3.43}$$

where κ and λ respectively follow from (3.34) and (3.35) with $\hat{\theta}_0$ replacing $\bar{\theta}_0$.

Since (2.49) is a fundamental matrix solution associated with A defined by (2.47) so is also the matrix

$$\hat{\Phi}(\theta) = \begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} & 1 \\ -\sin\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.44}$$

Analogous to (2.52) we define

$$\hat{R}(\theta) = \hat{\Phi}^{-1}(\theta)B \tag{3.45}$$

and substitute into (3.40) of the preceding theorem. Setting $s = \sin\hat{\theta}_0$, $c = \cos\hat{\theta}_0$ and $\beta_i = \lambda\alpha_i$, $i = 1, 2, 3$ we obtain

$$\begin{pmatrix} \frac{1}{2}(5-3s^2-3c) & \frac{1}{2}(5-3c) & \frac{1}{2}(5-3s^2-3c) \\ \frac{3}{2}s(c-1) & 0 & \frac{3}{2}s(1-c) \\ \frac{1}{2}(3-5c) & \frac{1}{2}(3c-5) & \frac{1}{2}(3-5c) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{3.46}$$

The region Z_f is defined by the conical set generated by the column vectors of the coefficient matrix. To find the region in terms of inequalities, we first solve (3.46), use the identity $c^2 = 1 - s^2$, and factor the denominators to obtain

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \frac{-1}{(5-3c)(1-c)} & \frac{1}{3(-s)(1-c)} & \frac{-1}{(5-3c)(1-c)} \\ \frac{-2(5c-3)}{(5-3c)^2(1-c)} & 0 & \frac{-2(3c^2-3c+2)}{(5-3c)^2(1-c)} \\ \frac{-1}{(5-3c)(1-c)} & \frac{-1}{3(-s)(1-c)} & \frac{-1}{(5-3c)(1-c)} \end{pmatrix} \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{3.47}$$

It follows from (3.27) and the definition of $\hat{\theta}_0$ that all of the denominators in this matrix are positive. Multiplying by the positive common denominators, employing (3.41) and the definitions of β_i, $i = 1, 2, 3$, we find

$$3(-s)z_{f_1} - (5-3c)z_{f_2} + 3(-s)z_{f_3} > 0 \tag{3.48}$$

$$(5c-3)z_{f_1} + (3c^2 - 3c + 2)z_{f_3} > 0 \tag{3.49}$$

$$3(-s)z_{f_1} + (5-3c)z_{f_2} - 3(-s)z_{f_3} > 0 \tag{3.50}$$

From these inequalities it is seen that the subregion of Z_f that admits three-impulse solutions described by (3.33)-(3.35) satisfies

$$z_{f_1} + z_{f_3} > \frac{5-3c}{-3s}|z_{f_2}| \tag{3.51}$$

$$(5c-3)z_{f_1}+(3c^2-3c+2)z_{f_3}>0. \tag{3.52}$$

The antipodal region that admits three-impulse solutions by (3.30)-(3.32) is found from the fact that ϕ is decreased by π from the preceding solutions, resulting in primer vectors having opposite signs. For this reason the left hand side of (3.40) changes sign. The consequence is that all inequalities reverse in (3.48)-(3.52). The two separate Z_f regions admitting three-impulse solutions therefore have symmetry with respect to the origin.

In the calculation of an optimal trajectory, the actual velocity increments $\Delta\mathbf{v}_i$ are calculated from the transformed velocity increments $\Delta\mathbf{V}_i$. Solving for these from (2.37) and (2.38) we get

$$\Delta v_{\theta_i}=\frac{h_i^3}{\mu r_i}\Delta V_{2i} \tag{3.53}$$

$$\Delta v_{r_i}=h_i\Delta V_{1i}+\frac{v_{ri}h_i^2}{\mu}\Delta V_{2i} \tag{3.54}$$

for $i=1,2,3$.

Example: Select $\theta_0=\pi/2$ and $\theta_f=3\pi/2$. From (3.25) and (3.26) we see that $\theta_1=\pi/2$, $\theta_2=\pi$ and $\theta_3=3\pi/2$. We shall use (3.33)-(3.35). If we use (3.30)-(3.32) instead we get a solution antipodal to this one. Using $\phi=\pi-2i\pi$ we obtain $\kappa=3/8$, $\lambda=4/5$ and $\hat{\theta}_0=-\pi/2$. Eq. (3.47) becomes

$$\beta_1=\frac{1}{5}z_{f_1}-\frac{1}{3}z_{f_2}+\frac{1}{5}z_{f_3}>0 \tag{3.55}$$

$$\beta_2=\frac{-6}{25}z_{f_1}+\frac{4}{25}z_{f_3}>0 \tag{3.56}$$

$$\beta_3=\frac{1}{5}z_{f_1}+\frac{1}{3}z_{f_2}+\frac{1}{5}z_{f_3}>0 \tag{3.57}$$

where (3.51) and (3.52) become

$$z_{f_1}+z_{f_3}>\frac{5}{3}|z_{f_2}|,\ \ -3z_{f_1}+2z_{f_3}>0. \tag{3.58}$$

For the antipodal subregion the inequalities in (3.58) are reversed.

We observe that

$$p_1(\theta)=-\frac{4}{5}\sin\theta,\ \ p_2(\theta)=-\frac{8}{5}(\cos\theta+\frac{3}{8}). \tag{3.59}$$

Substituting into (3.39) we find

$$\Delta\mathbf{V}_1=\alpha_1\begin{pmatrix}4/5\\3/5\end{pmatrix} \tag{3.60}$$

$$\Delta\mathbf{V}_2=\alpha_2\begin{pmatrix}0\\-1\end{pmatrix} \tag{3.61}$$

$$\Delta \mathbf{V}_3 = \alpha_3 \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix} \tag{3.62}$$

The antipodal solution reverses the directions of these velocity increments. The actual velocity increments follow from (3.53) and (3.54). The vector $\mathbf{z}_f$ is calculated from (2.50) and (2.57).

These calculations were performed for the specific boundary conditions $r_0 = 8000km$, $v_{r_0} = -0.831929km/sec$, $v_{\theta_0} = 7.487362km/sec$, and $r_f = 6545.455km$, $v_{r_f} = -0.679267km/sec$, $v_{\theta_f} = 7.471940km/sec$. The result is

$$\mathbf{z}_f = (-2.777777 \times 10^{-5} \;\; 0 \;\; 5.555555 \times 10^{-5})^T \, km^{-1}$$

which clearly satisfies (3.58) and shows that the optimal rendezvous for these boundary conditions requires three impulses. The three velocity increments are found to be: $\Delta v_{\theta_1} = -0.23538km/sec$, and $\Delta v_{r_1} = -0.287688km/sec$ at $\theta_1 = \pi/2$, $\Delta v_{\theta_2} = -1.313191km/sec$ and $\Delta v_{r_2} = -7.691605 \times 10^{-2} km/sec$ at $\theta_2 = \pi$ where $r(\theta_2)$ is approximately $7384km$, and $\Delta v_{\theta_3} = -0.193261km/sec$ and $\Delta v_{r_3} = 0.310108km/sec$ at $\theta_3 = 3\pi/2$. The fictitious nominal circular orbit has a radius of approximately $7272km$. Simulation of the optimal three-impulse rendezvous trajectory connecting the initial and terminal conditions and showing the application of the three velocity impulses is depicted in $Fig.\,9$. The maximum deviation of the rendezvous trajectory from the nominal circular orbit is approximately 10% of the nominal radius.

4 Two-Impulse Solutions

A two-impulse solution is a solution for which $k = 2$ in (2.58)-(2.63). Recalling that f is defined by (3.19) it follows that a non-degenerate two-impulse solution has exactly two zeros of f on the interval $\theta_0 \leq \theta \leq \theta_f$.

The two-impulse solutions can be classified as either non-stationary solutions or stationary solutions.

The non-stationary solutions are non-degenerate and fall naturally into several categories. We recall that $\bar{\theta} = \theta + \phi$ where ϕ is an arbitrary constant. The following definitions apply in general to optimal k-impulse solutions but are especially relevant for $k = 2$. These definitions are illuminated by Figures $10 - 17$.

An optimal solution is called a **non-stationary two-intersection solution** if $\theta_1 < \theta_2$, $f(\theta_1) = f(\theta_2) = 0$, $f'(\theta_1) \neq 0$, $f'(\theta_2) \neq 0$, $0 < \bar{\theta}_2 - \bar{\theta}_1 < 2\pi$, there are no other zeros of f on the interval $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$, and there is an integer i such that $\bar{\theta}_1 + \bar{\theta}_2 = 2\pi i$. If $0 < \bar{\theta}_0 < \pi$ then $i = 1$, if $-\pi < \bar{\theta}_0 < 0$ then $i = 0$. Primer vector loci for two-intersection solutions are depicted in Figures 10 and 11.

An optimal solution is called a **non-stationary four-intersection solution** if $\theta_1 < \theta_2$, $f(\theta_1) = f(\theta_2) = 0$, $0 < \bar{\theta}_2 - \bar{\theta}_1 < \pi$, and there is an integer i such that $i\pi < \bar{\theta}_1 < \bar{\theta}_2 < (i+1)\pi$. If $-\pi < \bar{\theta}_0 < \pi$ then $i = 0$ or $i = -1$. Primer vector loci for four-intersection solutions can be seen in Figures 12 and 13.

An optimal solution is called a **non-stationary three-intersection solution** if $\theta_1 < \theta_2$, $f(\theta_1) = f(\theta_2) = 0$, and either $f'(\theta_1) = 0$ or $f'(\theta_2) = 0$ but not both. Examples of primer vector loci for three intersection solutions are presented in Figures 14 and 15.

An optimal solution is called a **stationary two-intersection** solution if $\theta_1 < \theta_2$, $f(\theta_1) = f(\theta_2)$, $f'(\theta_1) = f'(\theta_2) = 0$, $\bar{\theta}_2 - \bar{\theta}_1 = \pi$ and there are no other zeros of f on the interval $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$. Primer vector loci of stationary two-intersection solutions are presented in Figure 16. If $\bar{\theta}_2 - \bar{\theta}_1 > \pi$ it is called a **stationary multi-impulse solution** and is degenerate. Primer vector loci of stationary multi-impulse solutions are depicted in Figure 17. An optimal multi-impulse solution may have arbitrarily many impulses if $\theta_f - \theta_0$ is sufficiently large but always has an equivalent one-impulse or two-impulse solution having the same cost as will be shown in the subsequent investigation of degenerate solutions.

The following theorem shows that there is a partition of the various categories of two-impulse solutions.

Theorem: An optimal non-degenerate two-impulse solution is one and only one of the following:

1. a stationary two-intersection solution
2. a non-stationary two-intersection solution
3. a non-stationary three-intersection solution
4. a non-stationary four-intersection solution.

Proof: Consider an optimal non-degenerate two-impulse solution having the primer vector given by (3.11) and (3.12). Forming the function (3.19) we obtain

$$f(\theta) = \lambda^2[\sin^2\bar{\theta} + 4(\cos\bar{\theta} - \kappa)^2 - 1.$$

Differentiating this function,

$$f'(\theta) = -2\lambda^2 \sin\bar{\theta}(3\cos\bar{\theta} - 4\kappa).$$

If the velocity impulses are at $\bar{\theta}_1$ and $\bar{\theta}_2$, then $f(\theta_1) = f(\theta_2) = 0$ where $\bar{\theta}_0 \leq \bar{\theta}_1 < \bar{\theta}_2 \leq \bar{\theta}_f$. Since all solutions considered are two-impulse solutions then $k = 2$ in (2.58)-(2.63) and there are no other zeroes of f on the interval $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$.

1) If $\bar{\theta}_1$ and $\bar{\theta}_2$ are integer multiples of π then $f'(\theta_1) = f'(\theta_2) = 0$ and $\bar{\theta}_2 - \bar{\theta}_1$ is an integer multiple of π. Since there are no other zeros on the interval $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$, then $\bar{\theta}_2 - \bar{\theta}_1 = \pi$. This solution satisfies the definition of a stationary two-intersection solution.

2) If neither $\bar{\theta}_1$ nor $\bar{\theta}_2$ are integer multiples of π and there is an integer i such that $\bar{\theta}_1 + \bar{\theta}_2 = 2\pi i$ then $0 < \bar{\theta}_2 - \bar{\theta}_1 < 2\pi$. If not, then $\bar{\theta}_2 - \bar{\theta}_1 \geq 2\pi$ and there is an integer j such that either $\bar{\theta}' = 2\pi j - \bar{\theta}_2$ or $\bar{\theta}' = 2\pi j - \bar{\theta}_1$. In either case $\bar{\theta}_1 < \bar{\theta}' < \bar{\theta}_2$ and $f(\bar{\theta}') = 0$, establishing more than two zeros of f on the interval $\theta_0 < \theta < \theta_f$.

We show now that $f'(\theta_1) \neq 0$ and $f'(\theta_2) \neq 0$. We argue by contradiction. Suppose $f'(\theta_1) = 0$. Since $\bar{\theta}_1$ is not an integer multiple of π it is necessary that $\kappa = \frac{3}{4}\cos\bar{\theta}_1$. Differentiating $f'(\theta)$ we find

$$f''(\theta) = -2\lambda^2(6\cos^2\bar{\theta} - 4\kappa\cos\bar{\theta} - 3).$$

Evaluating at $\bar{\theta}_1$ and substituting for κ into this expression we obtain

$$f''(\theta_1) = 6\lambda^2\sin^2\bar{\theta}_1.$$

This is clearly positive since $\bar{\theta}_1$ is not an integer multiple of π. Since $f(\theta_1) = 0$, then $f(\theta) > 0$ on an interval where $\bar{\theta} > \bar{\theta}_1$. This violates (2.63) establishing the contradiction. Similarly if we suppose $f'(\theta_2) = 0$ we find that $f(\theta) > 0$ on an interval where $\bar{\theta} < \bar{\theta}_2$, establishing the contradiction. As a consequence $f'(\theta_1) \neq 0$, $f'(\theta_2) \neq 0$, and the definition of a non-stationary two-intersection solution is satisfied.

3) If $\bar{\theta}_1$ is not an integer multiple of π but $\bar{\theta}_2$ is an integer multiple of π, clearly $f'(\theta_2) = 0$ and $f'(\theta_1) \neq 0$. If $f'(\theta_1) = 0$ the argument above establishes a contradiction. It follows that the definition of a non-stationary three-intersection solution is satisfied.

If $\bar{\theta}_1$ is an integer multiple of π but $\bar{\theta}_2$ is not an integer multiple of π, clearly $f'(\theta_1) = 0$ and the above argument shows that $f'(\theta_2) \neq 0$, so that in this case also, the definition of a non-stationary three-intersection solution is satisfied.

4) Finally, if neither $\bar{\theta}_1$ nor $\bar{\theta}_2$ are integer multiples of π and if $\bar{\theta}_1 + \bar{\theta}_2 \neq 2\pi i$ for any integer i then there is an integer i such that $i\pi < \bar{\theta}_1 < \bar{\theta}_2 < (i+1)\pi$. The reason is as follows. If this statement is not true and if i is the largest integer such that $i\pi < \bar{\theta}_1$ then $\bar{\theta}_2 \geq (i+1)\pi$. It follows then that there is an integer j such that if $\bar{\theta}' = 2\pi j - \bar{\theta}_1 \neq \bar{\theta}_2$ or $\bar{\theta}' = 2\pi j - \bar{\theta}_2 \neq \bar{\theta}_2$ then $\bar{\theta}_1 < \bar{\theta}' < \bar{\theta}_2$ and $f(\bar{\theta}') = 0$. This establishes at least three zeros on $\bar{\theta}_0 < \bar{\theta} < \bar{\theta}_f$ and a contradiction.

Having established that there is an integer i such that $i\pi < \bar{\theta}_1 < \bar{\theta}_2 < (i+1)\pi$ we subtract $\bar{\theta}_1$ in this inequality and obtain $0 < \bar{\theta}_2 - \bar{\theta}_1 < \pi$. The definition of a non-stationary four-intersection solution is therefore satisfied. ■

4.1 Non-Stationary Two-Intersection Solutions

First we consider the category of solutions in which the primer vector loci demonstrate the geometry represented in Figures 10 and 11, the non-tangential intersection with the unit circle. In the following we specify that $\theta_0 < \theta_f$. We define $\bar{\theta}_r = \theta_r - \phi$ for any subscript r.

4.1.1 Fundamental Theorem of Non-Stationary Two-Intersection Solutions

Theorem: The optimal impulsive rendezvous defined by (2.40) and (2.47) on the interval $\theta_0 \leq \theta \leq \theta_f$ satisfying the convention $-\pi \leq \bar{\theta}_0 \leq \pi$ has a non-stationary two-intersection solution $\{\theta_1, \theta_2, \Delta\mathbf{V}_1, \Delta\mathbf{V}_2\}$ if and only if

$$\theta_1 = \theta_0, \;\; \theta_2 = \theta_f, \tag{4.1}$$

$$0 < \theta_f - \theta_0 < 2\pi, \tag{4.2}$$

$$f(\theta) < 0, \quad \theta_0 < \theta < \theta_f, \tag{4.3}$$

$$f'(\theta_0) = -f'(\theta_f) < 0, \tag{4.4}$$

and either

$$0 < \bar{\theta}_0 < \pi, \;\; \bar{\theta}_f = 2\pi - \bar{\theta}_0, \;\; \mathit{and}\ \phi = \frac{\theta_0 + \theta_f}{2} - \pi, \tag{4.5}$$

or

$$-\pi < \bar{\theta}_0 < 0, \;\; \bar{\theta}_f = -\bar{\theta}_0, \;\; \mathit{and}\ \phi = \frac{\theta_0 + \theta_f}{2}, \tag{4.6}$$

and, in addition,

$$\Delta\mathbf{V}_1 = -\mathbf{p}(\theta_1)\alpha_1, \;\; \Delta\mathbf{V}_2 = -\mathbf{p}(\theta_2)\alpha_2, \tag{4.7}$$

$$R(\theta_1)\mathbf{p}(\theta_1)\alpha_1 + R(\theta_2)\mathbf{p}(\theta_2)\alpha_2 = -\mathbf{z}_f, \tag{4.8}$$

$$\alpha_1 > 0, \;\; \alpha_2 > 0. \tag{4.9}$$

Proof: First we show that (4.1)-(4.9) are necessary.

If the optimization problem has a non-stationary two-intersection solution, then, $0 < \bar{\theta}_2 - \bar{\theta}_1 < 2\pi$ where $f(\theta_1) = f(\theta_2) = 0$ and there is an integer i such that $\bar{\theta}_1 + \bar{\theta}_2 = 2\pi i$ and $\bar{\theta}_0 \leq \bar{\theta}_1 \leq \bar{\theta}_2 \leq \bar{\theta}_f$. Since $f'(\theta_1) \neq 0$ and $f'(\theta_2) \neq 0$, it follows from (2.62) that $\bar{\theta}_1 = \bar{\theta}_0$ and $\bar{\theta}_2 = \bar{\theta}_f$ and neither of these is an integer multiple of π. From (2.63) we have $f(\theta) \leq 0$, $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$. Since there are not more than two zeros of f on this interval we must have $f(\theta) < 0$, $\bar{\theta}_0 < \bar{\theta} < \bar{\theta}_f$. We have established (4.1) and (4.3); (4.4) follows from the continuity of f, the fact that there are only two zeros of f on the interval $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$, $f'(\theta_0) \neq 0$, $f'(\theta_f) \neq 0$, and $\bar{\theta}_1$ and $\bar{\theta}_2 = 2\pi i - \bar{\theta}_1$ can be substituted into the expression for $f'(\theta)$.

Since $\bar{\theta}_0$ is non-stationary we have two cases, either $0 < \bar{\theta}_0 < \pi$ or $-\pi < \bar{\theta}_0 < 0$.

In the former case we observe that $\bar{\theta}_m = 2\pi - \bar{\theta}_0$ is a zero of f satisfying $0 < \theta_m - \theta_0 < 2\pi$. This requires that $\bar{\theta}_f \leq \bar{\theta}_m$ and (4.2) is satisfied. Solving $\bar{\theta}_f = 2\pi - \bar{\theta}_0$ for ϕ we find that $\phi = \frac{\theta_0 + \theta_f}{2} - \pi$ establishing (4.5).

In the latter case we note that $\bar{\theta}_n = -\bar{\theta}_0$ is a zero of f satisfying $0 < \bar{\theta}_n - \bar{\theta}_0 < 2\pi$. This requires that $\bar{\theta}_f \leq \bar{\theta}_n$ and (4.2) is also satisfied in this case. Solving $\bar{\theta}_f = -\bar{\theta}_0$ for ϕ we find that $\phi = \frac{\theta_0 + \theta_f}{2}$ establishing (4.6) in this case.

The expressions (4.7)-(4.9) follow from the fact that an optimal solution satisfies (2.58)-(2.60) then (4.3) shows the solution is non-degenerate and the inequalities in (4.9) are strict.

To show that (4.1)-(4.9) are also sufficient we note that they imply (58)-(63) of Reference 1 with $k = 2$ showing optimality, where (4.3) shows that f has exactly two roots on $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$. The expressions (4.1),(4.2) and (4.4) show that $f'(\theta_1) \neq 0$ and $f'(\theta_2) \neq 0$; (4.1) and (4.5) or (4.6) show that $\bar{\theta}_1 + \bar{\theta}_2 = 2\pi i$ where $i = 1$ or $i = 0$.

These show that the optimal rendezvous is a non-stationary two-intersection solution completing the proof. ■

Remark: The convention $-\pi \leq \bar{\theta}_0 \leq \pi$ is convenient as an assumption in this theorem but unnecessary. See the previous theorem for three-impulse solutions. Without this assumption (4.5) and (4.6) would assert the existence of an integer i such that $2i\pi < \bar{\theta}_0 < (2i+1)\pi$ and $(2i-1)\pi < \bar{\theta}_0 < 2i\pi$ respectively.

4.1.2 Finding Non-Stationary Two-Intersection Solutions

Since $\lambda > 0$ we set $\beta_1 = \lambda\alpha_1$, $\beta_2 = \lambda\alpha_2$ and (4.9) becomes

$$\beta_1 > 0, \;\; \beta_2 > 0. \tag{4.10}$$

We observe that $\Phi(\bar{\theta})$ is a fundamental matrix solution associated with A so $R(\bar{\theta}_1)$ and $R(\bar{\theta}_2)$ may be inserted in (4.8). We select (4.6) so that $\bar{\theta}_f = -\bar{\theta}_0$ for the following development. There is no need to repeat this development for (4.5) because it leads to the antipodal solution.

Setting $s_0 = sin(\bar{\theta}_0)$ and $c_0 = cos(\bar{\theta}_0)$ (4.8) becomes

$$\begin{pmatrix} s_0^2 + 4c_0(c_0-\kappa) & s_0^2 + 4c_0(c_0-\kappa) \\ \\ -s_0c_0 + 4s_0(c_0-\kappa) & s_0c_0 - 4s_0(c_0-\kappa) \\ \\ -4c_0(c_0-\kappa) & -4c_0(c_0-\kappa) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{4.11}$$

For brevity we let

$$q_0 = 4(c_0 - \kappa) \tag{4.12}$$

and obtain the following three equations in the unknowns β_1, β_2 and q_0

$$(s_0^2 + c_0q_0)(\beta_1 + \beta_2) = -z_{f_1} \tag{4.13}$$

$$s_0(c_0 - q_0)(\beta_1 - \beta_2) = z_{f_2} \tag{4.14}$$

$$q_0(\beta_1 + \beta_2) = z_{f_3}. \tag{4.15}$$

Solving these equations we find that

$$\beta_1 = -\frac{1}{2}\left[\frac{z_{f_1}}{s_0^2 + c_0q_0} - \frac{z_{f_2}}{s_0(c_0 - q_0)}\right] \tag{4.16}$$

$$\beta_2 = -\frac{1}{2}\left[\frac{z_{f_1}}{s_0^2 + c_0 q_0} + \frac{z_{f_2}}{s_0(c_0 - q_0)}\right] \tag{4.17}$$

$$q_0 = -\frac{s_0^2 z_{f_3}}{z_{f_1} + c_0 z_{f_3}}. \tag{4.18}$$

These equations are subject to (4.10). The expression for κ is found from (4.12) and (4.18). Setting $f(\bar{\theta}_0) = 0$ we find

$$\lambda = 2(4s_0^2 + q_0^2)^{1/2}. \tag{4.19}$$

Substituting (4.18) into (4.16) and (4.17) we obtain

$$\beta_1 = -\frac{z_{f_1} + c_0 z_{f_3}}{2s_0^2}\left(1 - \frac{s_0 z_{f_2}}{c_0 z_{f_1} + z_{f_3}}\right) \tag{4.20}$$

$$\beta_2 = -\frac{z_{f_1} + c_0 z_{f_3}}{2s_0^2}\left(1 + \frac{s_0 z_{f_2}}{c_0 z_{f_1} + z_{f_3}}\right). \tag{4.21}$$

In view of the inequality (4.10) we find it necessary and sufficient for boundary conditions to satisfy

$$z_{f_1} + c_0 z_{f_3} < 0, \;\; |s_0 z_{f_2}| < |c_0 z_{f_1} + z_{f_3}| \tag{4.22}$$

to have non-stationary two-intersection solutions satisfying (4.6). For antipodal solutions satisfying (4.5) the necessary and sufficient conditions become

$$z_{f_1} + c_0 z_{f_3} > 0, \;\; |s_0 z_{f_2}| < |c_0 z_{f_1} + z_{f_3}|. \tag{4.23}$$

Example: Select $\bar{\theta}_0 = -\pi/2$, $\bar{\theta}_f = \pi/2$, $c_0 = 0$ and $s_0 = -1$. From (4.18) we find that $q_0 = z_{f_3}/z_{f_1}$; it follows from (4.12) that $\kappa = -\frac{z_{f_3}}{4z_{f_1}}$. We also find from (4.19) that $\lambda = 2(4 + \frac{z_{f_3}^2}{z_{f_1}^2})^{1/2}$ although this formula will not be needed.

For this example the region (4.22) of admissible solutions becomes

$$z_{f_1} < 0, \;\; |z_{f_2}| < |z_{f_3}|. \tag{4.24}$$

The expressions (4.20) and (4.21) respectively become

$$\beta_1 = -\frac{z_{f_1}}{2}\left(1 + \frac{z_{f_2}}{z_{f_3}}\right) \tag{4.25}$$

$$\beta_2 = -\frac{z_{f_1}}{2}\left(1 - \frac{z_{f_2}}{z_{f_3}}\right). \tag{4.26}$$

The optimal velocity increments therefore follow from (4.7)

$$\Delta\mathbf{V}_1 = \begin{pmatrix} -1 \\ -2\kappa \end{pmatrix}\beta_1, \;\; \Delta\mathbf{V}_2 = \begin{pmatrix} 1 \\ -2\kappa \end{pmatrix}\beta_2 \;\; . \tag{4.27}$$

The actual velocity increments are obtained from (3.53) and (3.54). For antipodal solutions the first inequality of (4.24) and the directions of (4.27) are reversed.

This example was simulated for $r_0 = 6545km$, $v_{r_0} = -0.784km/sec$, $\dot{\theta}_0 = 1.3\times10^{-3}rad/sec$, $r_f = 8000km$, $v_{r_f} = -0.710km/sec$, $\dot{\theta}_f = 0.798 \times 10^{-3}rad/sec$, boundary values which satisfy (23). The resulting optimal rendezvous orbit is depicted in Figure 18. The arrows in the figure indicate the application of the optimal velocity increments $\Delta V_1 = (\Delta V_x, \Delta V_y) = (0.4795, -0.8282)^T km/sec$, $\Delta V_2 = (\Delta V_x, \Delta V_y) = (-0.0142, 0.2986)^T km/sec$.

4.2 Non-Stationary Four-Intersection Solutions

Next we consider the category of solutions in which the primer vector loci demonstrate the geometry represented in Figures 12 and 13, the four intersections. Again we specify that $\theta_0 < \theta_f$.

4.2.1 Fundamental Theorem of Non-Stationary Four-Intersection Solutions

The following theorem is fundamental for this type of two-impulse solutions.

Theorem: The optimal impulsive rendezvous problem defined by (2.40) and (2.47) on the interval $\theta_0 \le \theta \le \theta_f$ satisfying the convention $-\pi \le \bar{\theta}_0 \le \pi$ has a non-stationary four-intersection solution $\{\theta_1, \theta_2, \Delta\mathbf{V}_1, \Delta\mathbf{V}_2\}$ if and only if

$$\theta_1 = \theta_0, \;\; \theta_2 = \theta_f, \tag{4.28}$$

$$either \;\; 0 < \bar{\theta}_0 < \bar{\theta}_f < \pi \quad or \quad -\pi < \bar{\theta}_0 < \bar{\theta}_f < 0, \tag{4.29}$$

and

$$f(\theta) < 0, \;\; \theta_0 \le \theta \le \theta_f, \tag{4.30}$$

$$f'(\theta_0) < 0, \;\; f'(\theta_f) > 0, \tag{4.31}$$

$$\kappa = \frac{3}{8}(\cos\bar{\theta}_0 + \cos\bar{\theta}_f), \;\; -\frac{3}{4} < \kappa < \frac{3}{4}, \tag{4.32}$$

$$\lambda = [\sin^2\bar{\theta}_0 + 4(\cos\bar{\theta}_0 - \kappa)^2]^{-1/2}, \;\; \frac{1}{2} < \frac{1}{2(1+|\kappa|)} \le \lambda \le (\frac{3}{3-4\kappa^2})^{1/2} < 2 \tag{4.33}$$

and (4.7)-(4.9) are satisfied.

Proof: First we show that (4.28)-(4.33) and (4.7)-(4.9) are necessary.

We suppose that the optimization problem has a non-stationary four-intersection solution. It follows that $f(\theta_1) = f(\theta_2) = 0$ where f is defined by (3.19), $\bar{\theta}_0 \le \bar{\theta}_1 < \bar{\theta}_2 \le \bar{\theta}_f$, there is an integer i such that $i\pi < \bar{\theta}_1 < \bar{\theta}_2 < (i+1)\pi$, $0 < \bar{\theta}_2 - \bar{\theta}_1 < \pi$, and $k = 2$ in (2.58)-(2.63) from the definition of non-stationary four-intersection solution. By (2.62) there are no other zeros of f on $\bar{\theta}_0 \le \theta \le \bar{\theta}_f$, hence $\bar{\theta}_1 = \bar{\theta}_0$ and $\bar{\theta}_2 = \bar{\theta}_f$. From the convention $-\pi \le \bar{\theta}_0 \le \pi$ it follows that either $i = 0$ and $0 < \bar{\theta}_0 < \bar{\theta}_f < \pi$ or $i = -1$ and $-\pi < \bar{\theta}_0 < \bar{\theta}_f < \pi$. Since $\bar{\theta}_0$

and $\bar{\theta}_f$ are the only zeros of f on $\bar{\theta}_0 \le \bar{\theta} \le \bar{\theta}_f$, (2.63) implies that $f(\theta) < 0$ on the interval $\bar{\theta}_0 < \bar{\theta} < \bar{\theta}_f$. We have established (2.28)-(2.30).

Using (3.11),(3.12) and (3.19) we have

$$f(\theta) = \lambda^2[\sin^2\bar{\theta} + 4(\cos\bar{\theta} - \kappa)^2] - 1$$

and

$$f'(\theta) = -2\lambda^2 \sin\bar{\theta}\,(3\cos\bar{\theta} - 4\kappa).$$

Setting $f(\theta_0) = f(\theta_f)$ we get

$$(\cos\bar{\theta}_f - \cos\bar{\theta}_0)[3(\cos\bar{\theta}_0 + \cos\bar{\theta}_f) - 8\kappa] = 0.$$

Since $0 < \bar{\theta}_f - \bar{\theta}_0 < \pi$ then $\cos\bar{\theta}_f \ne \cos\bar{\theta}_0$ establishing (4.32). If $f'(\theta_0) = 0$ then $\kappa = 3/4\cos\bar{\theta}_0$. Since

$$f''(\theta) = -2\lambda^2(6\cos^2\theta - 4\kappa\cos\bar{\theta} - 3)$$

then $f''(\theta_0) = 6\lambda^2\sin^2\bar{\theta}_0 > 0$ showing that $f(\theta) > 0$ on an interval where $\bar{\theta} > \bar{\theta}_0$ contradicting (2.63) and establishing that $f'(\theta_0) \ne 0$. A similar argument shows that $f'(\theta_f) \ne 0$. Either $f'(\theta_0) > 0$ or $f'(\theta_f) < 0$ implies that (4.30) is false establishing (2.31).

The above arguments have shown (4.28)-(4.32). Solving the equation $f(\theta_0) = 0$ for λ and minimizing and maximizing this expression with respect to θ_0 establishes (4.33). Finally (4.7)-(4.9) follow from (2.38) and (2.58)-(2.60) by setting $k = 2$.

We now show that (4.28)-(4.33) and (4.7)-(4.9) are sufficient. These conditions show that (2.58)-(2.63) are satisfied for $k = 2$ resulting in an optimal two-impulse solution of the problem. This solution is non-degenerate because (4.31) implies that θ_0 and θ_f are non-stationary. It is a four-intersection solution because (4.28) and (4.29) reveal that $i = -1$ or $i = 0$ and $0 < \bar{\theta}_2 - \bar{\theta}_1 < \pi$ follows from (4.29) after setting $\bar{\theta}_1 = \bar{\theta}_0$ and $\bar{\theta}_2 = \bar{\theta}_f$. ■

4.2.2 Finding Non-Stationary Four-Intersection Solutions

Since $\lambda > 0$ we again set $\beta_1 = \lambda\alpha_1$, $\beta_2 = \lambda\alpha_2$ and (4.9) must be satisfied. For brevity we set $c_0 = \cos\bar{\theta}_0$, $s_0 = \sin\bar{\theta}_0$, $c_f = \cos\bar{\theta}_f$ and $s_f = \sin\bar{\theta}_f$. Since $\Phi(\bar{\theta})$ is a fundamental matrix solution associated with A we again use $R(\bar{\theta}_1)$ and $R(\bar{\theta}_2)$ in (4.8). Setting $\bar{\theta}_1 = \bar{\theta}_0$, $\bar{\theta}_2 = \bar{\theta}_f$, substituting $\mathbf{p}(\theta_0)$ and $\mathbf{p}(\theta_f)$ into (4.8) and utilizing (4.32), after rearranging we obtain:

$$\begin{pmatrix} 1 + \frac{3}{2}c_0(c_0 - c_f) & 1 - \frac{3}{2}c_f(c_0 - c_f) \\ \\ \frac{3}{2}s_0(c_0 - c_f) & -\frac{3}{2}s_f(c_0 - c_f) \\ \\ -c_0 - \frac{3}{2}(c_0 - c_f) & -c_f + \frac{3}{2}(c_0 - c_f) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{4.34}$$

Solving these equations, we find that

$$\beta_1 = -\frac{s_f}{D} z_{f_1} - \frac{\frac{2}{3(c_0 - c_f)} - c_f}{D} z_{f_2} \tag{4.35}$$

$$\beta_2 = -\frac{s_0}{D} z_{f_1} + \frac{\frac{2}{3(c_0 - c_f)} + c_0}{D} z_{f_2} \tag{4.36}$$

if

$$z_{f_3} = \frac{[9(s_0 c_0 + s_f c_f) - 15(c_0 s_f + c_f s_0)]}{6D} z_{f_1} - \frac{[9c_0^2 - 30 c_0 c_f + 9 c_f^2 + 16]}{6D} z_{f_2} \tag{4.37}$$

where

$$D = s_0 + s_f + \frac{3}{2}(c_0 - c_f)(c_0 s_f - c_f s_0). \tag{4.38}$$

The velocity increments from (4.7) become

$$\Delta \mathbf{V}_1 = \begin{pmatrix} -s_0 \\ -\frac{5}{4}c_0 + \frac{3}{4}c_f \end{pmatrix} \beta_1, \quad \Delta \mathbf{V}_2 = \begin{pmatrix} -s_f \\ \frac{3}{4}c_0 - \frac{5}{4}c_f \end{pmatrix} \beta_2. \tag{4.39}$$

In the case where $0 < \bar{\theta}_0 < \pi$, then $D > 0$ and a non-degenerate four-intersection solution is found only in a $\mathbf{z}_f$ region where

$$s_f z_{f_1} + \left[\frac{2}{3(c_0 - c_f)} - c_f\right] z_{f_2} < 0, \quad s_0 z_{f_1} - \left[\frac{2}{3(c_0 - c_f)} + c_0\right] z_{f_2} < 0, \tag{4.40}$$

and z_{f_3} satisfies (4.37). Geometrically this says that there are solutions restricted to a sector described by (4.40) of the plane described by (4.37).

In the case where $-\pi < \bar{\theta}_0 < 0$ then $D < 0$ and the inequalities in (4.40) are reversed but (4.37) remains valid. Geometrically this reverses the sector in the plane with every vector $\mathbf{z}_f$ in the sector being replaced by $-\mathbf{z}_f$. This is the antipodal case.

Of some interest is the special case where $\kappa = 0$ so that $c_0 = -c_f$, $s_0 = s_f$ resulting in a primer locus having symmetry about a vertical line through the origin. The equations (4.35)-(4.37) simplify:

$$\beta_1 = -\frac{z_{f_1}}{2(1 + 3c_0^2)} - \frac{z_{f_2}}{6 s_0 c_0} \tag{4.41}$$

$$\beta_2 = -\frac{z_{f_1}}{2(1 + 3c_0^2)} + \frac{z_{f_2}}{6 s_0 c_0} \tag{4.42}$$

$$z_{f_3} = -\frac{4}{3 s_0} z_{f_2}. \tag{4.43}$$

The velocity increments (4.39) also simplify somewhat in this case. This type of solution exists only for the $\mathbf{z}_f$ region where

$$z_{f_2} < -\frac{3 s_0 c_0}{1 + 3c_0^2} z_{f_1}, \quad z_{f_2} > \frac{3 s_0 c_0}{1 + 3c_0^2} z_{f_1} \tag{4.44}$$

and z_{f_3} satisfies (4.43), consequently

$$z_{f_1} < 0, \quad \frac{3s_0c_0}{1+3c_0^2}z_{f_1} < z_{f_2} < -\frac{3s_0c_0}{1+3c_0^2}z_{f_1} \tag{4.45}$$

and

$$\frac{4s_0c_0}{1+3c_0^2}z_{f_1} < s_0z_{f_3} < -\frac{4s_0c_0}{1+3c_0^2}z_{f_1}. \tag{4.46}$$

Example: Select $\bar{\theta}_0 = \pi/6$ and $\bar{\theta}_f = 5\pi/6$, then $c_0 = -c_f = \sqrt{3}/2$, $s_0 = s_f = 1/2$, $\beta_1 = -2/13z_{f_1} - 2\sqrt{3}/9z_{f_2}$, $\beta_2 = -2/13z_{f_1} + 2\sqrt{3}/9z_{f_2}$ and

$$\mathbf{\Delta V}_1 = \begin{pmatrix} -\frac{1}{2} \\ -\sqrt{3} \end{pmatrix}\beta_1, \quad \mathbf{\Delta V}_2 \begin{pmatrix} -\frac{1}{2} \\ \sqrt{3} \end{pmatrix}\beta_2. \tag{4.47}$$

We select values of y_0 and y_f so that $z_{f_1} < 0$ and $z_{f_2} = z_{f_3} = 0$. These conditions are satisfied by the boundary values $r_0 = 8000km$, $v_{r_0} = 0km/sec$, $\dot{\theta}_0 = 8.37 \times 10^{-4}rad/sec$, $r_f = 6545km$, $v_{r_f} = -0.866km/sec$, $\dot{\theta}_f = 1.251 \times 10^{-3}rad/sec$. Simulation of this example with these boundary values was performed, and the optimal rendezvous orbit is presented in Figure 19. The imposition of the velocity increments $\Delta V_1 = (\Delta V_x, \Delta V_y) = (0.0382, -0.1216)^T km/sec$, $\Delta V_2 = (\Delta V_x, \Delta V_y) = (-0.0362, -0.1007)^T km/sec$ is indicated by the arrows.

4.3 Two-Impulse Non-Stationary Three-intersection Solutions

Three-impulse non-stationary three-intersection solutions are discussed in Sec. 3.3.2. If θ_f is somewhat decreased or θ_0 increased from that of a non-degenerate three-impulse solution as shown in Figures 14 and 15 the result is a two-impulse non-stationary three-intersection solution.

4.3.1 Fundamental Theorem of Two-impulse Non-Stationary Three-intersection Solutions

Theorem: The optimal impulsive rendezvous problem defined by (2.40) and (2.47) on the interval $\theta_0 \le \theta \le \theta_f$ satisfying the convention $-\pi \le \bar{\theta}_0 \le \pi$ has a two-impulse non-stationary three-intersection solution $\{\theta_1, \theta_2, \mathbf{\Delta V}_1, \mathbf{\Delta V}_2\}$ if and only if $0 < \bar{\theta}_0 < \pi$ and

$$\bar{\theta}_1 = \bar{\theta}_0,\, \bar{\theta}_2 = \pi \le \bar{\theta}_f < 2\pi - \bar{\theta}_0,\, f'(\theta_0) < 0,\, \kappa = \frac{3}{8}(\cos\bar{\theta}_0 - 1),\, \lambda = \frac{4}{5+3\cos\bar{\theta}_0}, \tag{4.48}$$

or

$$2\pi - \bar{\theta}_f < \bar{\theta}_0 < \bar{\theta}_1 = \pi,\, \bar{\theta}_2 = \bar{\theta}_f,\, f'(\theta_f) > 0,\, \kappa = \frac{3}{8}(\cos\bar{\theta}_f - 1),\, \lambda = \frac{4}{5+3\cos\bar{\theta}_f}, \tag{4.49}$$

or else $-\pi < \bar{\theta}_0 < 0$ and

$$\bar{\theta}_1 = \bar{\theta}_0,\ \bar{\theta}_2 = 0 \leq \bar{\theta}_f < -\bar{\theta}_0,\ f'(\theta_0) < 0,\ \kappa = \frac{3}{8}(\cos\bar{\theta}_0 + 1),\ \lambda = \frac{4}{5 - 3\cos\bar{\theta}_0}, \tag{4.50}$$

or

$$-\pi < -\bar{\theta}_f < \bar{\theta}_0 < \bar{\theta}_1 = 0,\ \bar{\theta}_2 = \bar{\theta}_f,\ f'(\theta_f) > 0,\ \kappa = \frac{3}{8}(\cos\bar{\theta}_f + 1),\ \lambda = \frac{4}{5 - 3\cos\bar{\theta}_f}, \tag{4.51}$$

and in either case

$$f(\theta) \leq 0, \bar{\theta}_0 < \bar{\theta} < \bar{\theta}_f, \tag{4.52}$$

and (4.7)-(4.9) are satisfied.

Proof: First we show that (4.48)-(4.52) and (4.7)-(4.9) are necessary.

If the optimization problem has a two-impulse non-stationary three-intersection solution then $f(\theta_1) = f(\theta_2) = 0$ where $\bar{\theta}_0 \leq \bar{\theta}_1 < \bar{\theta}_2 \leq \bar{\theta}_f$ and either $f'(\theta_1) = 0$ or $f'(\theta_2) = 0$a but not both, and $-\pi \leq \bar{\theta}_0 \leq \pi$. We observe that

$$f(\theta) = \lambda^2[\sin^2\bar{\theta} + 4(\cos\bar{\theta} - \kappa)^2] - 1$$

$$f'(\theta) = -2\lambda^2 \sin\bar{\theta}(3\cos\bar{\theta} - 4\kappa)$$

$$f''(\theta) = -2\lambda^2(6\cos^2\bar{\theta} - 4\kappa\cos\bar{\theta} - 3)$$

Equating $f(\theta_1)$ and $f(\theta_2)$ and performing some manipulations we obtain

$$(\cos\bar{\theta}_2 - \cos\bar{\theta}_1)[3(\cos\bar{\theta}_1 + \cos\bar{\theta}_2) - 8\kappa] = 0.$$

We shall show that $\cos\bar{\theta}_1 \neq \cos\bar{\theta}_2$.

Suppose that $f'(\theta_1) \neq 0$ and $f'(\theta_2) = 0$. If these are reversed the following argument is also valid with cyclic interchange of θ_1 and θ_2.

Setting $f'(\theta_2) = 0$ implies either $\sin\bar{\theta}_2 = 0$ or $\kappa = \frac{3}{4}\cos\bar{\theta}_2$ but the latter implies $f''(\theta_2) = 6\lambda^2\sin^2\bar{\theta}_2 > 0$, which implies that $f(\theta) > 0$ on an interval where $\bar{\theta} < \bar{\theta}_2$ because $f(\theta_2) = 0$, contradicting (2.63). It follows that $\sin\bar{\theta}_2 = 0$. If $\cos\bar{\theta}_1 = \cos\bar{\theta}_2$ then $\sin\bar{\theta}_1 = 0$ also setting $f'(\theta_1) = 0$ contrary to the original supposition consequently $\kappa = \frac{3}{8}(\cos\bar{\theta}_1 + \cos\bar{\theta}_2)$.

Continuing $f'(\theta_2) = 0$ implies $\sin\bar{\theta}_2 = 0$ which implies that $\bar{\theta}_2 = 0$ or $\bar{\theta}_2 = \pi$.

If $\bar{\theta}_2 = \pi$ then $0 < \bar{\theta}_1 < \bar{\theta}_2$ since $f'(\theta_1) \neq 0$. Setting $k = 2$ in (2.62), we see that $\bar{\theta}_1 = \bar{\theta}_0$, so $0 < \bar{\theta}_0 < \pi$, $\bar{\theta}_2 \leq \theta_f$, and $\bar{\theta}_f < 2\pi - \bar{\theta}_0$ because $f(2\pi - \bar{\theta}_0) = 0$. Clearly $f'(\theta_0) < 0$ because $f'(\theta_0) > 0$ with $f(\theta_0) = 0$ establishes an interval where $\bar{\theta} > \bar{\theta}_0$ and (2.63) is violated. Substituting $\cos\bar{\theta}_2 = -1$ in the above expression for κ reveals $\kappa = \frac{3}{8}(\cos\bar{\theta}_0 - 1)$. The expression for λ comes from setting $f(\theta_2) = 0$ substituting for κ and solving for λ. We have established (4.48).

If $\bar{\theta}_2 = 0$ the proof is analogous; $-\pi < \bar{\theta}_1 < \bar{\theta}_2$ since $f(\theta_1) \neq 0$. Again setting $k = 2$ in (2.62) we get $\bar{\theta}_1 = \bar{\theta}_0$ so $-\pi < \bar{\theta}_0 < 0$, $\bar{\theta}_2 \leq \bar{\theta}_f$ and $\bar{\theta}_f < -\bar{\theta}_0$. Again (2.63) implies

that $f'(\theta_0) < 0$. Substituting $\cos\bar{\theta}_2 = 1$ into the above expression for κ reveals that $\kappa = \frac{3}{8}(\cos\bar{\theta}_0 + 1)$. Again setting $f(\theta_2) = 0$ substituting for κ and evaluating λ completes (4.50).

The expressions (4.49) and (4.51) follow from $f'(\theta_1) = 0$ and $f(\theta_2) \neq 0$ by repeating the preceding argument with θ_1 and θ_2 interchanged. These arguments lead to an interchange of $\bar{\theta}_0$ and $\bar{\theta}_f$ and a reversal of all inequalities.

The expression (4.52) is a restatement of (2.63) and (4.7)-(4.9) follow from (2.58)-(2.60) by setting $k = 2$. The inequalities in (4.9) are strict because the solution is a non-degenerate two-impulse solution.

To show that (4.48)-(4.52) and (4.7)-(4.9) are sufficient we note that they imply (2.58)-(2.63) with $k = 2$ resulting in an optimal two-impulse solution. In each of (4.48)-(4.51) $\bar{\theta}_0 \leq \bar{\theta}_1 < \bar{\theta}_2 \leq \bar{\theta}_f$, and $f(\theta_1) = f(\theta_2) = 0$ and either $f'(\theta_1) \neq 0$ or $f'(\theta_2) \neq 0$ but not both, hence the solution is a non-stationary three-intersection solution and not degenerate. ∎

4.3.2 Finding Two-Impulse Non-Stationary Three-Intersection Solutions

First we consider the case where $0 < \bar{\theta}_0 < \pi$, $\bar{\theta}_0 = \bar{\theta}_1 < \bar{\theta}_2 = \pi < \bar{\theta}_f < 2\pi - \bar{\theta}_0$. Again we set $c_0 = \cos\bar{\theta}_0$, $s_0 = \sin\bar{\theta}_0$ and observe that $\Phi(\bar{\theta})$ is a fundamental matrix solution associated with A. Substituting $\kappa = \frac{3}{8}(c_0 - 1)$ into $\mathbf{p}(\theta_0)$ and $\mathbf{p}(\theta_2)$ the expression (4.8) becomes

$$\begin{pmatrix} \frac{1}{2}(3c_0^2 + 3c_0 + 2) & \frac{1}{2}(3c_0 + 5) \\ \frac{3}{2}s_0(c_0 + 1) & 0 \\ -\frac{1}{2}(5c_0 + 3) & \frac{1}{2}(3c_0 + 5) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} z_{f_1} \\ z_{f_2} \\ z_{f_3} \end{pmatrix}. \tag{4.53}$$

A solution exists if

$$\beta_1 = \frac{2z_{f_2}}{3s_0(c_0 + 1)}, \tag{4.54}$$

$$\beta_2 = \frac{6s_0(c_0 + 1)z_{f_1} - 2(3c_0^2 + 3c_0 + 2)z_{f_2}}{3s_0(c_0 + 1)(3c_0 + 5)}, \tag{4.55}$$

and

$$z_{f_3} = z_{f_1} - \frac{(3c_0 + 5)z_{f_2}}{3s_0}. \tag{4.56}$$

It is necessary from (4.9) that $\beta_1 > 0$ and $\beta_2 > 0$, and since the denominators in (4.54) and (4.55) must be positive, it is necessary that the $\mathbf{z}_f$ region satisfy (4.56) and

$$z_{f_1} > \frac{(3c_0^2 + 3c_0 + 2)z_{f_2}}{3s_0(c_0 + 1)}, \quad z_{f_2} > 0. \tag{4.57}$$

Geometrically, this region is a sector of a plane through the origin (56). If $\mathbf{z}_f$ is contained in this sector then (4.7) determines optimal velocity increments

$$\Delta\mathbf{V}_1 = \begin{pmatrix} s_0 \\ \frac{5c_0+3}{4} \end{pmatrix} \beta_1, \quad \Delta\mathbf{V}_2 = -\begin{pmatrix} 0 \\ \frac{3c_0+5}{4} \end{pmatrix} \beta_2. \tag{4.58}$$

We note also that if $-\pi < \bar{\theta}_0 < 0$, $\bar{\theta}_0 = \bar{\theta}_1 < \bar{\theta}_2 = 0 < \bar{\theta}_f < -\bar{\theta}_0$ then we have an antipodal solution where $\mathbf{z}_f$ is replaced by $-\mathbf{z}_f$ in (4.53)-(4.57).

Next we consider the case where $2\pi - \bar{\theta}_f < \bar{\theta}_0 < \bar{\theta}_1 = \pi$, $\bar{\theta}_2 = \bar{\theta}_f$. We set $c_f = \cos\bar{\theta}_f$, $s_f = \sin\bar{\theta}_f$ and substitute $\kappa = \frac{3}{8}(c_f - 1)$ in $\mathbf{p}(\theta_1)$ and $\mathbf{p}(\theta_f)$. The expression (4.8) becomes

$$\begin{pmatrix} \frac{1}{2}(3c_f+5) & \frac{1}{2}(3c_f^2+3c_f+2) \\ 0 & \frac{3}{2}s_f(c_f+1) \\ \frac{1}{2}(3c_f+5) & -\frac{1}{2}(5c_f+3) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{4.59}$$

A solution exists if

$$\beta_1 = \frac{-6s_f(c_f+1)z_{f_1} + 2(3c_f^2+3c_f+2)z_{f_2}}{3s_f(c_f+1)(3c_f+5)}, \tag{4.60}$$

$$\beta_2 = \frac{-2z_{f_2}}{3s_f(c_f+1)}, \tag{4.61}$$

and

$$z_{f_3} = z_{f_1} - \frac{3c_f+5}{3s_f} z_{f_2}. \tag{4.62}$$

Since $\beta_1 > 0$, $\beta_2 > 0$, and the denominators in (4.60) and (4.61) are negative, The $\mathbf{z}_f$ region must satisfy (2.61) and

$$z_{f_1} < \frac{3c_f^2+3c_f+2}{3s_f(c_f+1)} z_{f_2}, \quad z_{f_2} > 0. \tag{4.63}$$

Geometrically this is a sector of the plane (4.62). If $\mathbf{z}_f$ is contained in this sector, then optimal velocity increments are determined by (4.7)

$$\Delta\mathbf{V}_1 = \begin{pmatrix} 0 \\ \frac{3c_f+5}{4} \end{pmatrix} \beta_1, \quad \Delta\mathbf{V}_2 = -\begin{pmatrix} s_f \\ \frac{5c_f+3}{4} \end{pmatrix} \beta_2. \tag{4.64}$$

We note that if $-\pi < \bar{\theta}_0 < 0$, $-\bar{\theta}_f < \bar{\theta}_0 < \bar{\theta}_1 = 0$, $\bar{\theta}_2 = \bar{\theta}_f$ then we have an antipodal solution with $\mathbf{z}_f$ replaced by $-\mathbf{z}_f$ in (4.59)-(4.63).

Example: Select $\bar{\theta}_0 = \pi/2$, $\bar{\theta}_2 = \pi$ and $\bar{\theta}_f = \frac{4\pi}{3}$. It follows that $c_0 = 0$, $s_0 = 1$, $c_f = -\frac{1}{2}$ $s_f = -\frac{\sqrt{3}}{2}$, (4.57) becomes

$$z_{f_1} > \frac{2}{3} z_{f_2}, \quad z_{f_2} > 0 \tag{4.65}$$

and (4.56) becomes

$$z_{f_3} = z_{f_1} - \frac{5}{3} z_{f_2}. \tag{4.66}$$

Substituting into (57) of Reference 1 we obtain

$$z_{f_1} = -\frac{1}{2} y_{f_1} + \frac{\sqrt{3}}{2} y_{f_2} + \frac{1}{2} y_{f_3} + y_{0_2}, \tag{4.67}$$

$$z_{f_2} = -\frac{\sqrt{3}}{2} y_{f_1} - \frac{1}{2} y_{f_2} + \frac{\sqrt{3}}{2} y_{f_3} - y_{0_1} + y_{0_3}, \tag{4.68}$$

$$z_{f_3} = y_{f_3} - y_{0_3}. \tag{4.69}$$

Combining (2.64)-(2.68), we find

$$(2\sqrt{3} - 6) y_{f_1} + (3\sqrt{3} + 2) y_{f_2} + (-2\sqrt{3} + 3) y_{f_3} + 4 y_{0_1} + 6 y_{0_2} - 4 y_{0_3} > 0, \tag{4.70}$$

$$-\sqrt{3} y_{f_1} - y_{f_2} + \sqrt{3} y_{f_3} - 2 y_{0_1} + 2 y_{0_3} > 0, \tag{4.71}$$

$$(3 - 5\sqrt{3}) y_{f_1} - (3\sqrt{3} + 5) y_{f_2} + (5\sqrt{3} + 5) y_{f_3} - 10 y_{0_1} - 6 y_{0_2} + 4 y_{0_3} = 0. \tag{4.72}$$

These inequalities are satisfied by the boundary values $r_0 = 7200km$, $v_{r_0} = 0km/sec$, $\dot{\theta}_0 = 1.033 \times 10^{-3} rad/sec$, $r_f = 10375km$, $v_{r_f} = -3.547km/sec$, $\dot{\theta}_f = 4.746 \times 10^{-4} rad/sec$. This example was simulated for these boundary values, and the resulting optimal rendezvous orbit is presented in Figure 20. The optimal velocity increments, represented by the arrows on the figure are $\Delta V_1 = (\Delta V_x, \Delta V_y) = (-0.3759, 0.5269)^T km/sec$, $\Delta V_2 = (\Delta V_x, \Delta V_y) = (-1.691, 0.9413)^T km/sec$.

4.4 Non-Degenerate Stationary Two-Impulse Solutions

A k-impulse solution is called **stationary** if $f(\theta_i) = 0$, $i = 1, \ldots, k$ for distinct θ_i on the interval $\theta_0 \leq \theta \leq \theta_f$ implies $f'(\theta_i) = 0$, $i = 1, \ldots, k$. It is called **non-degenerate** if there is no l-impulse solution for $l < k$. It can be shown from the periodicity of (2.60) that stationary solutions for $k > 2$ are degenerate.

Primer vector loci of stationary solutions are presented in Figures 16 and 17. Figure 16 depicts a primer vector locus for a stationary non-degenerate two-impulse solution, whereas Figure 17 represents a primer vector locus of a stationary degenerate multi-impulse solution.

Degenerate solutions are presented in a paper that follows.

4.4.1 Fundamental Theorem of Non-Degenerate Stationary Two-Impulse Solutions

We shall accept the convention $-\pi < \bar{\theta}_0 \leq \pi$. If $\bar{\theta}_0$ is outside this interval, a simple change of variable can be employed.

Theorem: The optimal impulsive rendezvous defined by (2.40) and (2.47) on the interval $\bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f$ satisfying the convention $-\pi < \bar{\theta}_0 \leq \pi$ has a non-degenerate stationary two-impulse solution $\{\theta_1, \theta_2, \Delta\mathbf{V}_1, \Delta\mathbf{V}_2\}$ if and only if

$$\kappa = 0, \lambda = \frac{1}{2} \tag{4.73}$$

$$\pi \leq \theta_f - \theta_0 < 3\pi, \tag{4.74}$$

and if $-\pi < \bar{\theta}_0 \leq 0$ then

$$\bar{\theta}_1 = 0,\ \ \bar{\theta}_2 = \pi,\ \ \phi = \theta_1 = \theta_2 - \pi,\ \ \pi \leq \bar{\theta}_f < 2\pi, \tag{4.75}$$

but if $0 < \bar{\theta}_0 \leq \pi$ then

$$\bar{\theta}_1 = \pi,\ \ \bar{\theta}_2 = 2\pi,\ \ \phi = \theta_1 - \pi = \theta_2 - 2\pi,\ \ 2\pi \leq \bar{\theta}_f, \tag{4.76}$$

and in either case

$$f(\theta) = -\frac{3}{4}\sin^2\bar{\theta} \leq 0,\ \ \bar{\theta}_0 \leq \bar{\theta} \leq \bar{\theta}_f \tag{4.77}$$

and (4.7)-(4.9) are satisfied.

Proof: First we show that (4.73)-(4.77) and (4.7)-(4.9) are necessary. It is shown in (3.19) that a stationary solution must be supported on integer multiples of π; non-degeneracy requires this support at the two smallest multiples of π greater than or equal $\bar{\theta}_0$, consequently $\bar{\theta}_f$ must be placed after the second smallest multiple of π and previous to the third. The results (4.74)- (4.76) follow and (4.73) is found from (3.20) and (3.21). Substituting (4.73) into the expression for the primer vector, we have

$$f(\theta) = \frac{1}{4}\sin^2\bar{\theta} + \cos^2\bar{\theta} - 1$$

which results in (4.77). We obtain (4.7)-(4.9) by setting $k = 2$ in (2.58)-(2.60); since the solution is non-degenerate the inequalities in (4.9) must be strict.

It is seen that (4.73)-(4.77) and (4.7)-(4.9) are sufficient because they show that this solution is stationary and (2.58)-(2.63) are satisfied for $k = 2$. This solution is non-degenerate because the inequalities in (4.9) are strict. ■

We now apply this theorem in order to find non-degenerate stationary solutions. We investigate two cases, $-\pi < \bar{\theta}_0 \leq 0$ and $0 < \bar{\theta}_0 \leq \pi$.

4.4.2 Finding Non-Degenerate Stationary Solutions

Case 1: Suppose $-\pi < \bar{\theta}_0 \leq 0$ then $\bar{\theta}_1 = 0$, $\bar{\theta}_2 = \pi$ and $\pi \leq \bar{\theta}_f < 2\pi$. Observing that $\mathbf{p}(\theta_1)^T = (0, 1)$ and $\mathbf{p}(\theta_2)^T = (0, -1)$, using $R(\bar{\theta})$ from (2.52), we see that (4.8) becomes

$$\begin{pmatrix} 2 & 2 \\ \\ 0 & 0 \\ \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{4.78}$$

The solution of this system of equations is

$$\alpha_1 = -\frac{1}{4}(z_{f_1} - z_{f_3}) \tag{4.79}$$

$$\alpha_2 = -\frac{1}{4}(z_{f_1} + z_{f_3}) \tag{4.80}$$

where

$$z_{f_2} = 0. \tag{4.81}$$

From (4.9), it is necessary that

$$z_{f_1} < -|z_{f_3}|. \tag{4.82}$$

Optimal two-impulse solutions for this case are found for boundary conditions satisfying (4.81) and (4.82). Geometrically the $\mathbf{z}_f$ region is restricted to a sector (4.82) of the plane (4.81). If these boundary conditions are satisfied, the optimal velocity increments are applied at $\bar{\theta}_1 = 0$, $\bar{\theta}_2 = \pi$ and are calculated from (4.7):

$$\Delta\mathbf{V}_1 = \begin{pmatrix} 0 \\ -\alpha_1 \end{pmatrix}, \quad \Delta\mathbf{V}_2 = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix}. \tag{4.83}$$

Case 2: Suppose $0 < \bar{\theta}_0 \leq \pi$ then $\bar{\theta}_1 = \pi$, $\bar{\theta}_2 = 2\pi$ and $2\pi \leq \bar{\theta}_f < 3\pi$. We note that $\mathbf{p}(\theta_1)^T = (0, -1)$ and $\mathbf{p}(\theta_2)^T = (0, 1)$. In this case (4.8) becomes

$$\begin{pmatrix} 2 & 2 \\ \\ 0 & 0 \\ \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -z_{f_1} \\ -z_{f_2} \\ -z_{f_3} \end{pmatrix}. \tag{4.84}$$

The solution of this system of equations is

$$\alpha_1 = -\frac{1}{4}(z_{f_1} + z_{f_3}) \tag{4.85}$$

$$\alpha_2 = -\frac{1}{4}(z_{f_1} - z_{f_3}) \tag{4.86}$$

where again (4.81) is satisfied. From (4.9), it is necessary also for this case that (4.82) is satisfied.

Optimal two-impulse solutions for this case also follow from the boundary conditions satisfying (4.81) and (4.82). For these boundary conditions the optimal velocity increments are applied at $\bar{\theta}_1 = \pi$, and $\bar{\theta}_2 = 2\pi$ and are calculated from (4.7):

$$\Delta\mathbf{V}_1 = \begin{pmatrix} 0 \\ \alpha_1 \end{pmatrix}, \quad \Delta\mathbf{V}_2 = \begin{pmatrix} 0 \\ -\alpha_2 \end{pmatrix}. \tag{4.87}$$

Example: Select $\bar{\theta}_0 = 0$, $\bar{\theta}_f = \pi$ so that $\bar{\theta}_1 = \bar{\theta}_0$, $\bar{\theta}_2 = \bar{\theta}_f$ and $\phi = \theta_0 = \theta_f - \pi$. Using (2.50) and (2.56) with $\bar{\theta}$ in (4.50) we have

$$z_{f_1} = -y_{f_1} + y_{f_3} - y_{0_1} + y_{0_3}$$

$$z_{f_2} = -y_{f_2} - y_{0_2} = 0$$

$$z_{f_3} = y_{f_3} - y_{0_3}.$$

The condition (4.81) becomes

$$y_{0_2} + y_{f_2} = 0$$

and (4.82) becomes

$$-y_{f_1} + y_{f_3} - y_{0_1} + y_{0_3} < -|y_{f_3} - y_{0_3}|.$$

If $y_{f_3} > y_{0_3}$ this becomes

$$2y_{f_3} < y_{0_1} + y_{f_1},$$

but if $y_{f_3} < y_{0_3}$ it becomes

$$2y_{0_3} < y_{0_1} + y_{f_1},$$

and if $y_{f_3} = y_{0_3}$ then

$$z_{f_1} = 2y_{0_3} - (y_{0_1} + y_{f_1}) < 0.$$

The boundary values $r_0 = 7200km$, $v_{r_0} = 0km/sec$, $\dot{\theta}_0 = 1.1558\times10^{-3}rad/sec$, $r_f = 7200km$, $v_{r_f} = 0km/sec$, $\dot{\theta}_f = 1.1554\times10^{-3}rad/sec.$ satisfy the first of the last three inequalities, and are used for simulation of this example. The resulting optimal rendezvous is shown in Figure 21. The optimal velocity increments $\Delta V_1 = (\Delta V_x, \Delta V_y) = (1.7756\times10^{-3}, -0.8779)^T km/sec$, $\Delta V_2 = (\Delta V_x, \Delta V_y) = (-4.3162 \times 10^{-4}, -0.8038)^T km/sec$ are indicated by arrows in the figure.

5 One-Impulse Solutions

For certain boundary conditions a minimizing solution may contain only one impulse. This situation is investigated here.

5.1 Fundamental Theorem of Non-Degenerate One-Impulse Solutions

We apply (2.58)-(2.63) where $k = 1$. If a one-impulse minimizing solution $\{\theta_1, \Delta\mathbf{V}_1\}$ is non-degenerate then $\Delta\mathbf{V}_1 \neq 0$. For this case applications are facilitated without use of the vector $\mathbf{z}_f$.

Theorem: The optimal impulsive rendezvous problem defined by (2.40) and (2.47) has a non-degenerate one-impulse solution $\{\theta_1, \Delta\mathbf{V}_1\}$ if and only if

$$\theta_1 = \theta_0 \textit{ or } f'(\theta_1) = 0 \textit{ or } \theta_1 = \theta_f, \tag{5.1}$$

$$f(\theta_1) = 0 \textit{ and } f(\theta) \leq 0 \textit{ on } \theta_0 \leq \theta \leq \theta_f, \tag{5.2}$$

$$\alpha_1 = |\Delta\mathbf{V}_1|, \tag{5.3}$$

$$\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} = -p(\theta_1), \tag{5.4}$$

$$B\Delta\mathbf{V}_1 = \Phi(\theta_1 - \theta_f)\mathbf{y}_f - \Phi(\theta_1 - \theta_0)\mathbf{y}_0. \tag{5.5}$$

Proof: Setting $k = 1$ and $\Delta\mathbf{V}_1 \neq 0$ in (2.58), (2.59) and (2.61)-(2.63), these expressions become equivalent to (5.1)-(5.4). In (2.60) we replace θ_1 by $\bar{\theta}_1$ obtaining $R(\bar{\theta}_1)\Delta\mathbf{V}_1 = \mathbf{z}_f$. Multiplying on the left by $\Phi(\bar{\theta}_1)$ in view of (2.52) the expression (5.5) emerges. ■

5.2 Finding Non-Degenerate One-Impulse Solutions

The expression (5.1) is resolved into two situations, the non-stationary solutions where $\theta_1 = \theta_0$ or $\theta_1 = \theta_f$, or the stationary solutions where $f'(\theta_1) = 0$.

5.2.1 Non-Stationary Solutions

In the following we set $\theta_d = \theta_f - \theta_0$, $c_d = \cos\theta_d$, and $s_d = \sin\theta_d$.

Case 1: Suppose $\theta_1 = \theta_0$ then (5.5) becomes

$$B\Delta\mathbf{V}_1 = \Phi(-\theta_d)\mathbf{y}_f - \mathbf{y}_0. \tag{5.6}$$

Utilizing (2.40) and (2.41) and solving (5.6) we obtain

$$\Delta V_{11} = -s_d y_{f_1} - c_d y_{f_2} + y_{0_2}, \tag{5.7}$$

$$\Delta V_{12} = -\frac{1}{2}(y_{f_3} - y_{0_3}), \tag{5.8}$$

where

$$y_{0_1} = c_d y_{f_1} - s_d y_{f_2} + y_{f_3}. \tag{5.9}$$

Expressions (5.7)-(5.8) are necessary and sufficient to have one-impulse solutions supported at θ_0.

Case 2: Suppose $\theta_1 = \theta_f$ then (5.5) becomes

$$B\Delta\mathbf{V}_1 = \mathbf{y}_f - \Phi(\theta_d)\mathbf{y}_0. \tag{5.10}$$

Solving,

$$\Delta V_{11} = -y_{f_2} - s_d y_{0_1} + c_d y_{0_2}, \tag{5.11}$$

$$\Delta V_{12} = -\frac{1}{2}(y_{f_3} - y_{0_3}), \tag{5.12}$$

where

$$y_{f_1} = c_d y_{0_1} + s_d y_{0_2} + y_{0_3}. \tag{5.13}$$

Expressions (5.11)-(5.13) are necessary and sufficient to have one-impulse solutions supported at θ_f.

5.2.2 Stationary Solutions

In the following we set $c_0 = \cos\bar{\theta}_0$, $s_0 = \sin\bar{\theta}_0$, $c_f = \cos\bar{\theta}_f$, and $s_f = \sin\bar{\theta}_f$.

Case 1: Suppose $\bar{\theta}_1 = 0$ and $-\pi < \bar{\theta}_0 < 0 < \bar{\theta}_f < \pi$, then (5.5) becomes

$$B\Delta\mathbf{V}_1 = \Phi(-\bar{\theta}_f)\mathbf{y}_f - \Phi(-\bar{\theta}_0)\mathbf{y}_0. \tag{5.14}$$

Solving, noting (5.4) and the fact that $p_1(\theta_1) = 0$ if $\bar{\theta}_1$ is a stationary point, we obtain

$$\Delta V_{11} = 0, \tag{5.15}$$

$$\Delta V_{12} = -\frac{1}{2}(y_{f_3} - y_{0_3}), \tag{5.16}$$

where

$$c_0 y_{0_1} - s_0 y_{0_2} + y_{0_3} = c_f y_{f_1} - s_f y_{f_2} + y_{f_3}, \tag{5.17}$$

$$s_0 y_{0_1} + c_0 y_{0_2} = s_f y_{f_1} + c_f y_{f_2}. \tag{5.18}$$

Expressions (5.15)-(5.18) are necessary and sufficient to have stationary one-impulse solutions at $\bar{\theta}_1 = 0$.

Case 2: Suppose $\bar{\theta}_1 = \pi$ and $0 < \bar{\theta}_0 < \pi < \bar{\theta}_f < 2\pi$, then (5.5) becomes

$$B\Delta\mathbf{V}_1 = \Phi(\pi - \bar{\theta}_f)\mathbf{y}_f - \Phi(\pi - \bar{\theta}_0)\mathbf{y}_0. \tag{5.19}$$

Solving, as before we find

$$\Delta V_{11} = 0, \tag{5.20}$$

$$\Delta V_{12} = -\frac{1}{2}(y_{f_3} - y_{0_3}), \tag{5.21}$$

where

$$c_0 y_{0_1} - s_0 y_{0_2} - y_{0_3} = c_f y_{f_1} - s_f y_{f_2} - y_{f_3}, \tag{5.22}$$

$$s_0 y_{0_1} - c_0 y_{0_2} = s_f y_{f_1} - c_f y_{f_2}. \tag{5.23}$$

Expressions (5.20)-(5.23) are necessary and sufficient to have stationary one-impulse solutions at $\bar{\theta}_1 = \pi$.

Conclusions

A planar impulsive rendezvous problem can be solved for initial and terminal positions and velocities near those associated with a nominal circular orbit. The rendezvous trajectories produced are exact for this restricted two-body problem and approach optimality as the boundary conditions approach boundary conditions of a nominal circular orbit.

By replacing the cost function (2.11) by the related cost function (2.48), it was found that velocity impulses that minimize (2.48) produce a close approximation of a minimum of (2.11) if the boundary conditions are near those of a nominal circular orbit.

The boundary conditions are incorporated into a vector $\mathbf{z}_f$ defined by (2.57) which determines the number of velocity impulses required and from which these velocity increments can be calculated. It was found that a non-degenerate rendezvous trajectory requires three impulses if and only if the boundary conditions are such that $\mathbf{z}_f$ or $-\mathbf{z}_f$ satisfies (3.51) and (3.52). It was found that the proper placement of the velocity impulses is at the initial, terminal, and mid-point values of the true anomaly. The calculations are simple enough that the three velocity increments could be found from a hand calculator that contains trigonometric functions, if necessary. An example of a three-impulse rendezvous problem with boundary conditions that lead to quick solution was presented. Simulation of the rendezvous trajectory that resulted showed that throughout the flight the deviation of the radial distance remained within ten percent of the radius of a nominal circular orbit.

It was found that non-degenerate two-impulse solutions of an optimal impulsive rendezvous near a circular orbit can be classified according to the ways the locus of a primer vector can intersect the unit circle. These optimal two-impulse solutions fall into four categories.

For each of these categories necessary and sufficient conditions for an optimal solution were presented. Distinct regions of the boundary conditions that admit optimal two-impulse solutions for each category were displayed. A closed-form solution of the optimal velocity increments for each category can be found. Some examples and simulations were presented.

Necessary and sufficient conditions for optimal non-degenerate one-impulse solutions were also found. These optimal one-impulse solutions consist of four types. Distinct regions of

boundary conditions admitting the optimal one-impulse solutions were also displayed. For each type a closed-form optimal velocity increment can be found.

Although this work emphasizes non-degenerate rendezvous in the vicinity of a circular orbit, it also presents a framework for studies of stationary, degenerate, singular rendezvous and optimal rendezvous beyond the vicinity of a circular orbit. Presentation of results of these studies is to follow. In these results we will show that degenerate and singular solutions are more than interesting curiosities. Not only do they provide important understanding to the subject of impulsive rendezvous, but they also comprise important solutions to some impulsive rendezvous problems.

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Fig. 19 - Rendezvous orbit (red) of a four-intersection solution $\bar{\theta}_0 = \pi/6$, $\bar{\theta}_f = 5\pi/6$, $r_0 = 8000km$, $v_{r_0} = 0km/sec$, $\dot{\theta}_0 = 8.37 \times 10^{-4} rad/sec$ $r_f = 6545km$, $v_{r_f} = -0.866km/sec$, $\dot{\theta}_f = 1.25 \times 10^{-3} rad/sec.$
Orbits satisfying initial conditions (green) and terminal conditions(blue) are displayed.
Fig. 20 - Rendezvous orbit (red) of a two impulse nonstationary three-intersection solution $\bar{\theta}_0 = \pi/2$, $\bar{\theta}_2 = \pi$, $\bar{\theta}_f = 4\pi/3$, $r_0 = 7200km$, $v_{r_0} = 0km/sec$, $\dot{\theta}_0 = 1.033 \times 10^{-3} rad/sec$ $r_f = 10375km$, $v_{r_f} = -3.547km/sec$, $\dot{\theta}_f = 4.746 \times 10^{-3} rad/sec.$
Orbits satisfying initial conditions (green) and terminal conditions(blue) are displayed.
Fig. 21 - Rendezvous orbit (red) of a nondegenerate stationary solution $\bar{\theta}_0 = 0$, $\bar{\theta}_f = \pi$, $r_0 = 7200km$, $v_{r_0} = 0km/sec$, $\dot{\theta}_0 = 1.1558 \times 10^{-3} rad/sec$

$r_f = 7200km$, $v_{r_f} = 0km/sec$, $\dot{\theta}_f = 1.1556 \times 10^{-3} rad/sec$.
Orbits satisfying initial conditions (green) and terminal conditions(blue) are displayed.

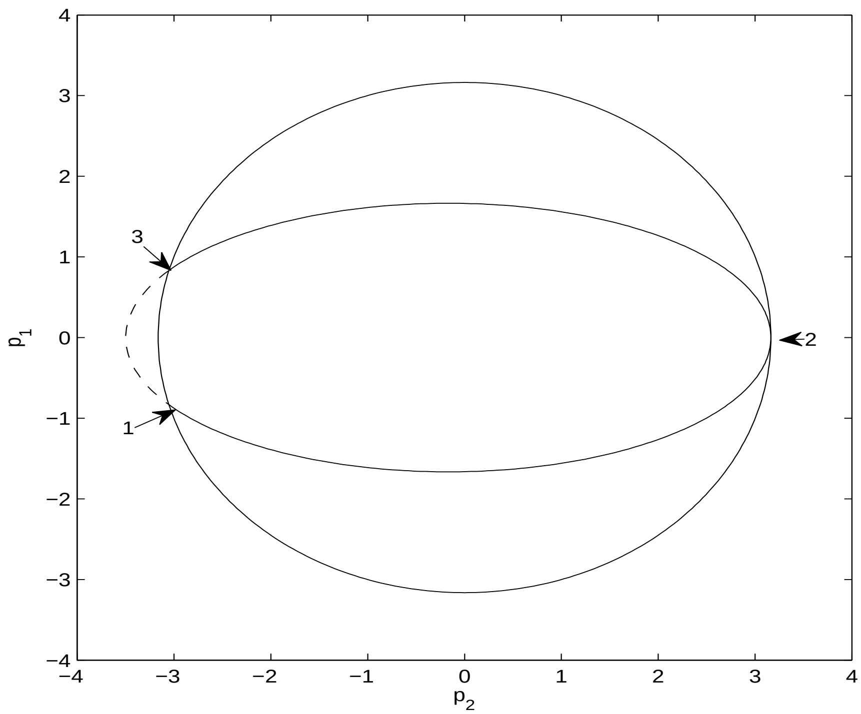
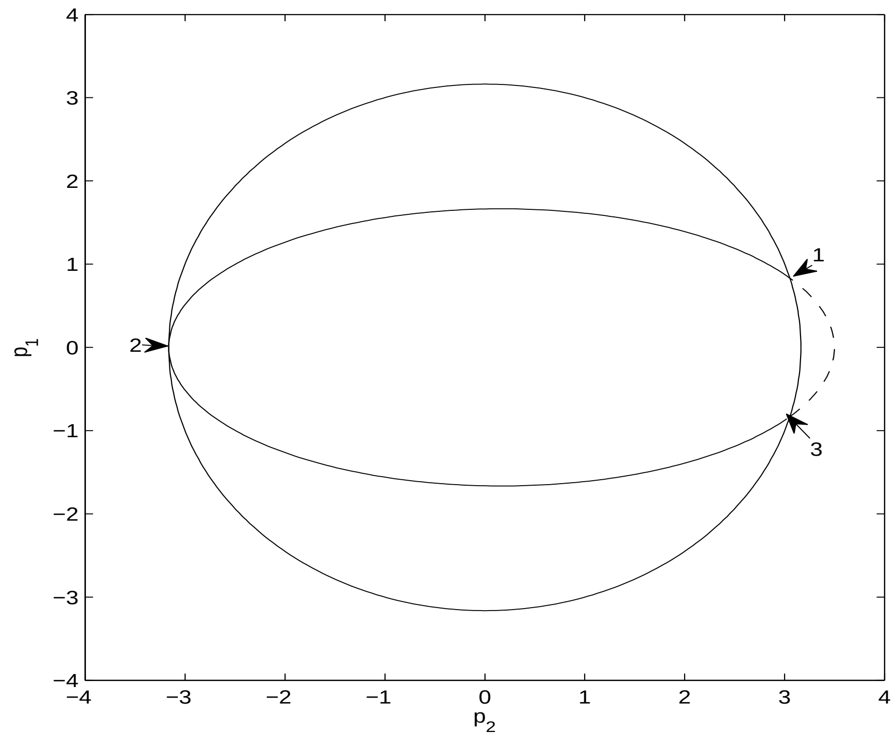


Figure 1:



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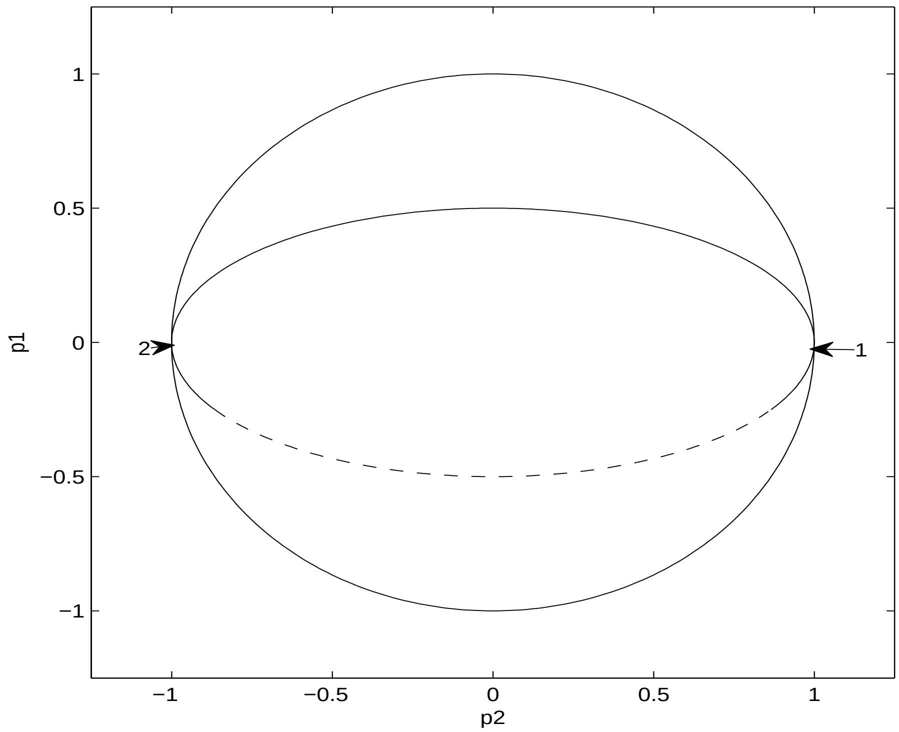

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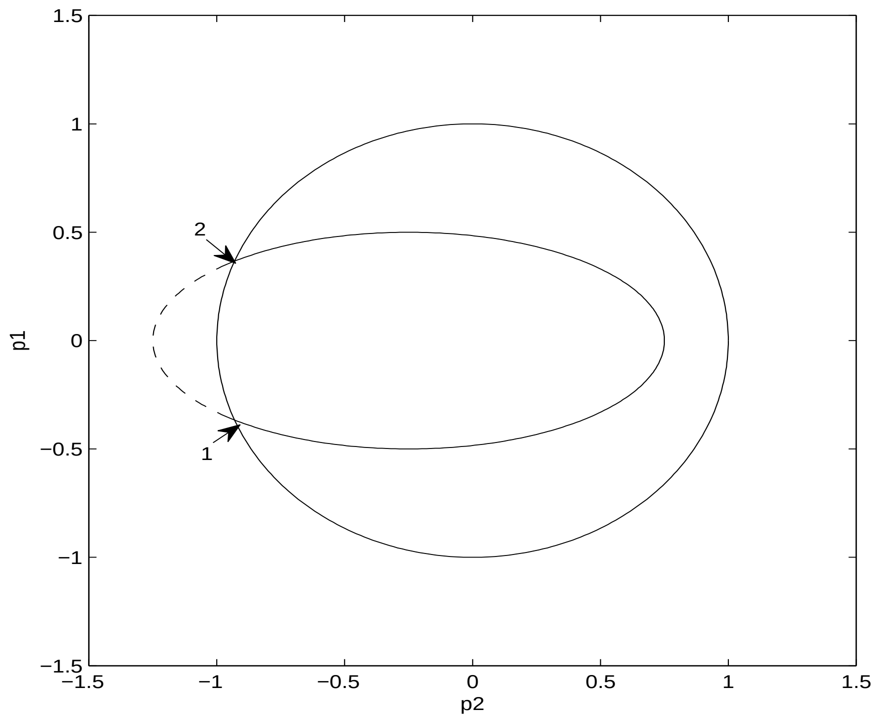

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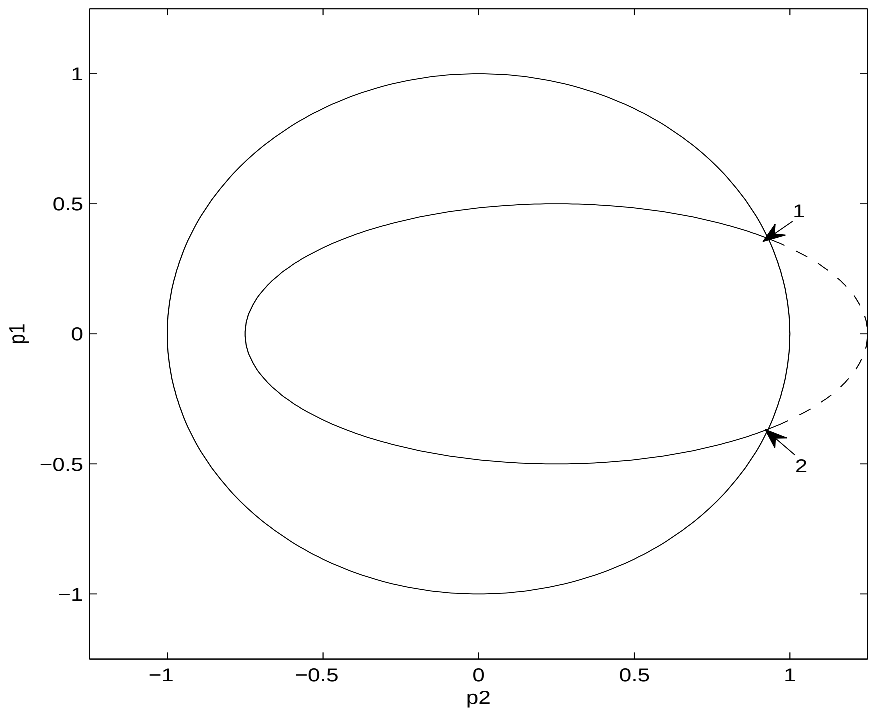

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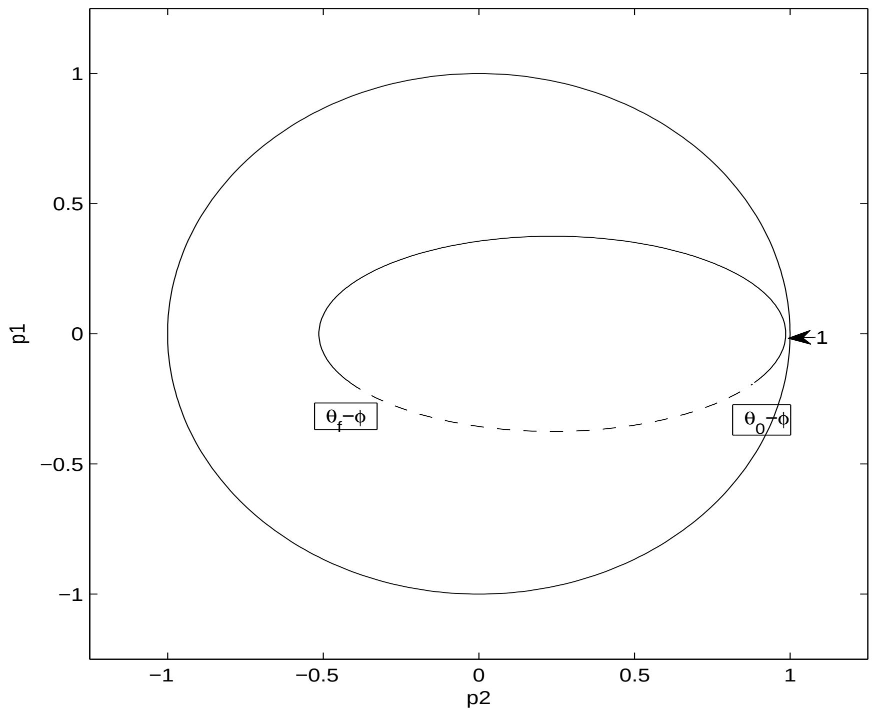

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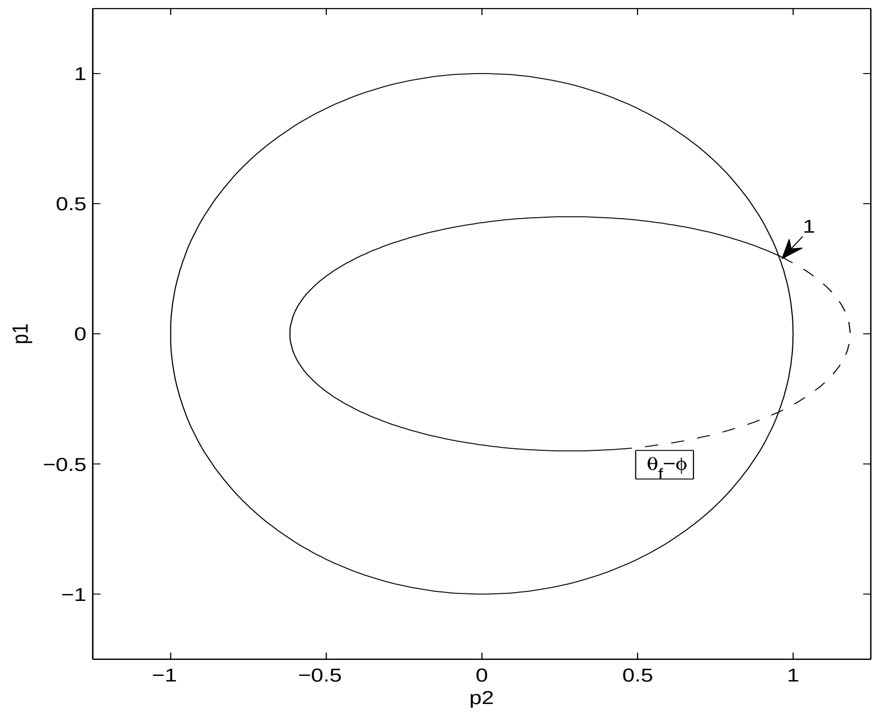

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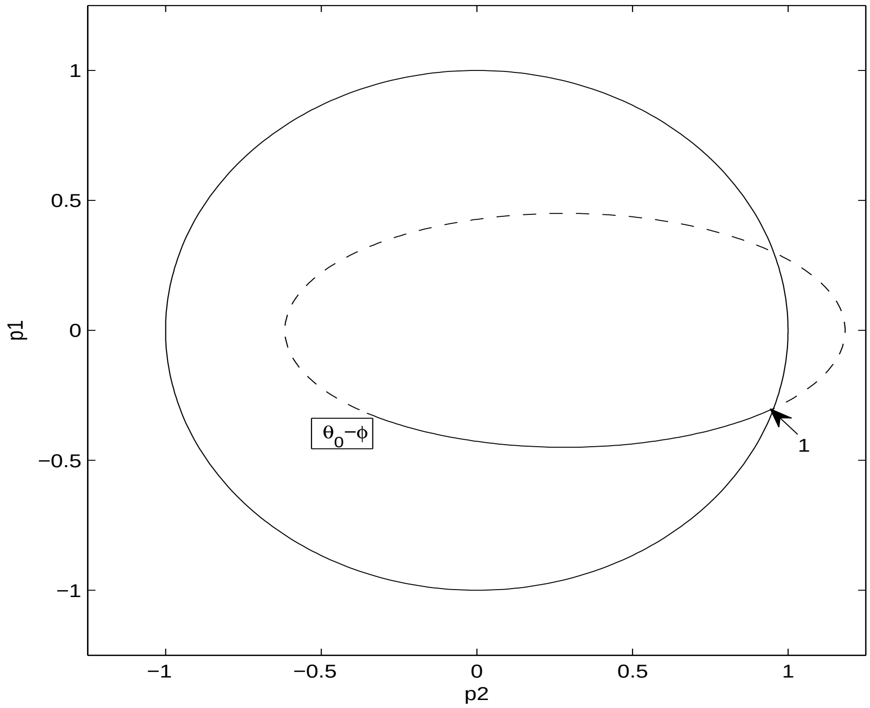

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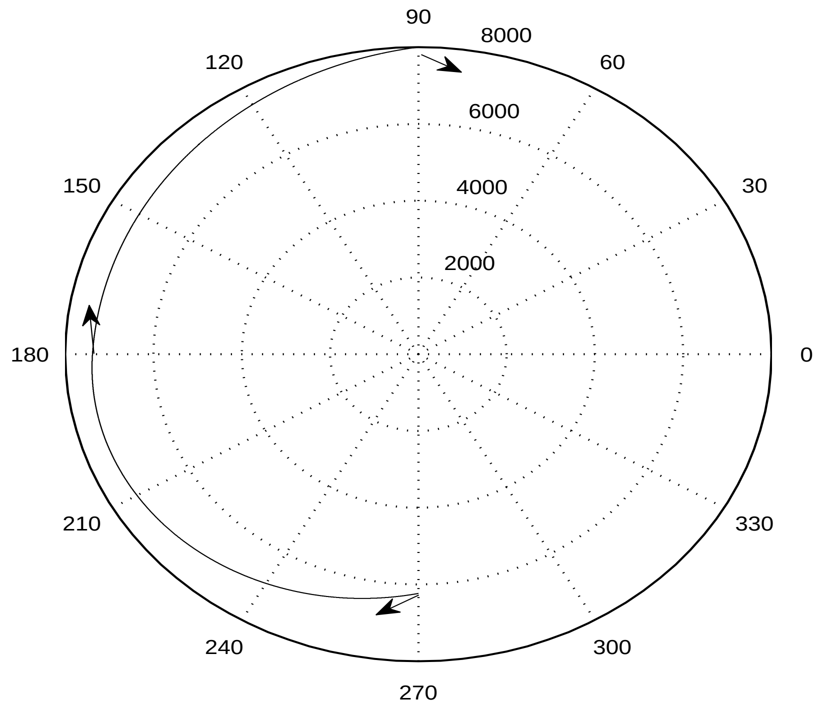

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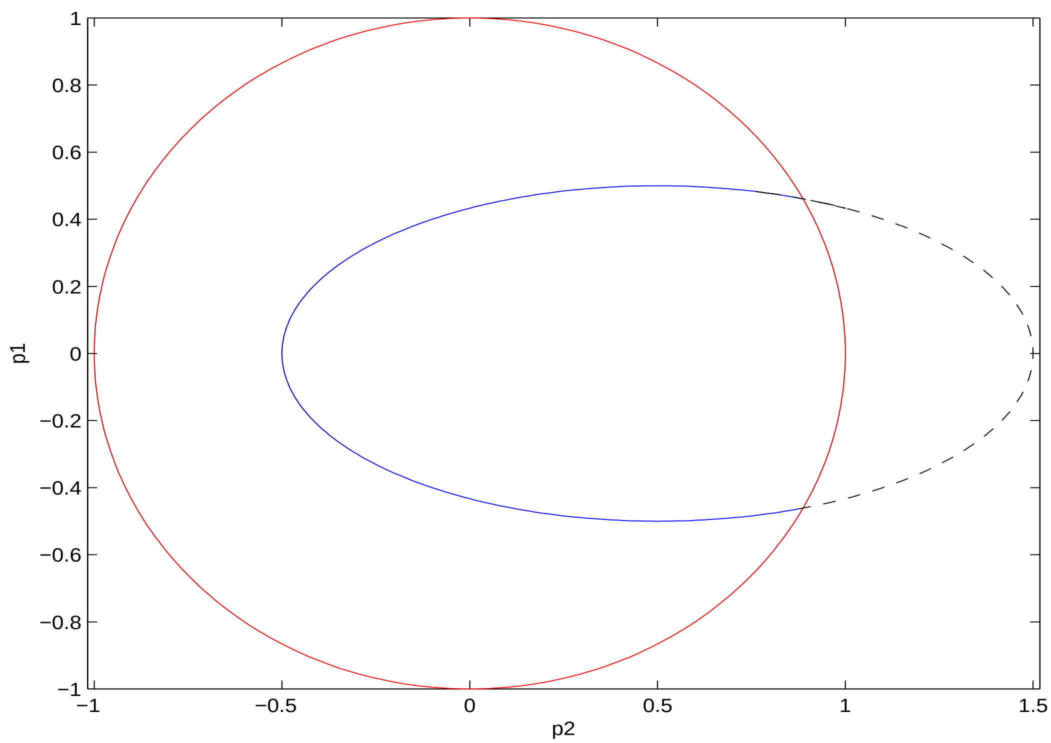

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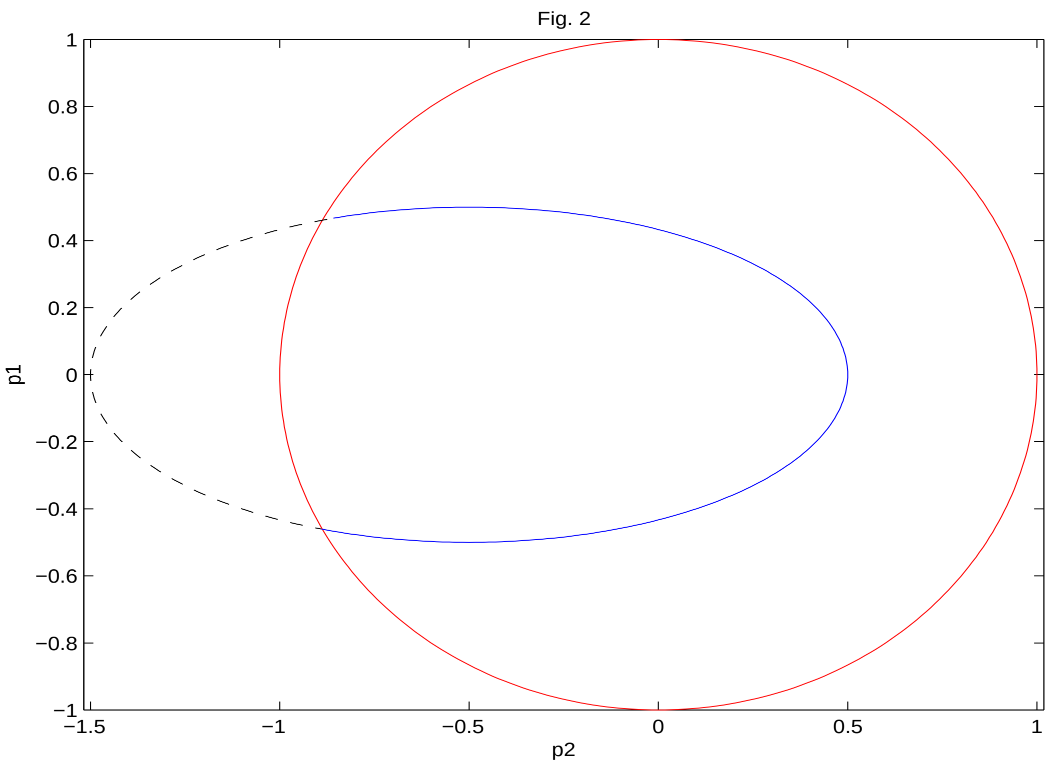

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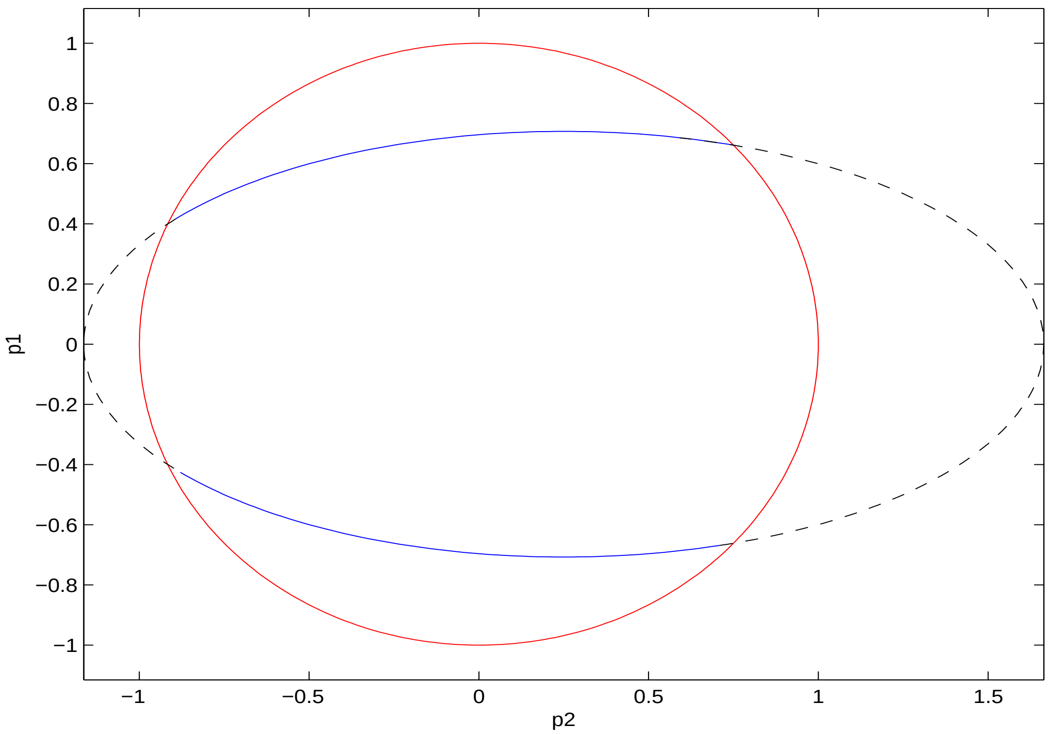

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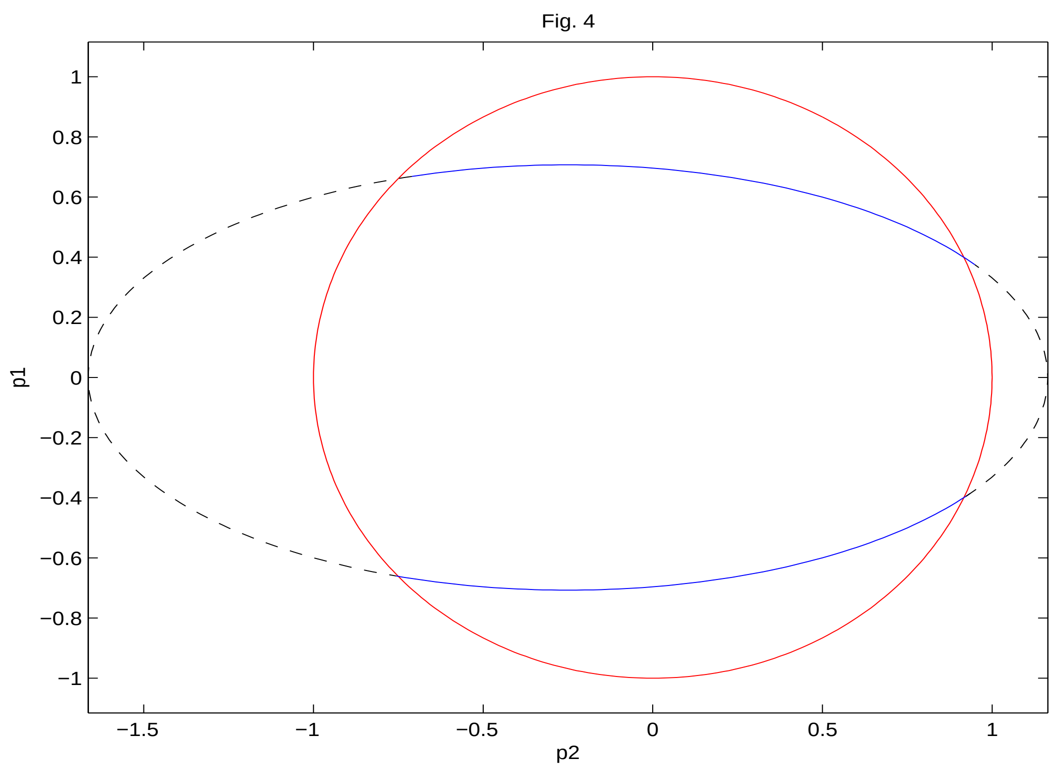

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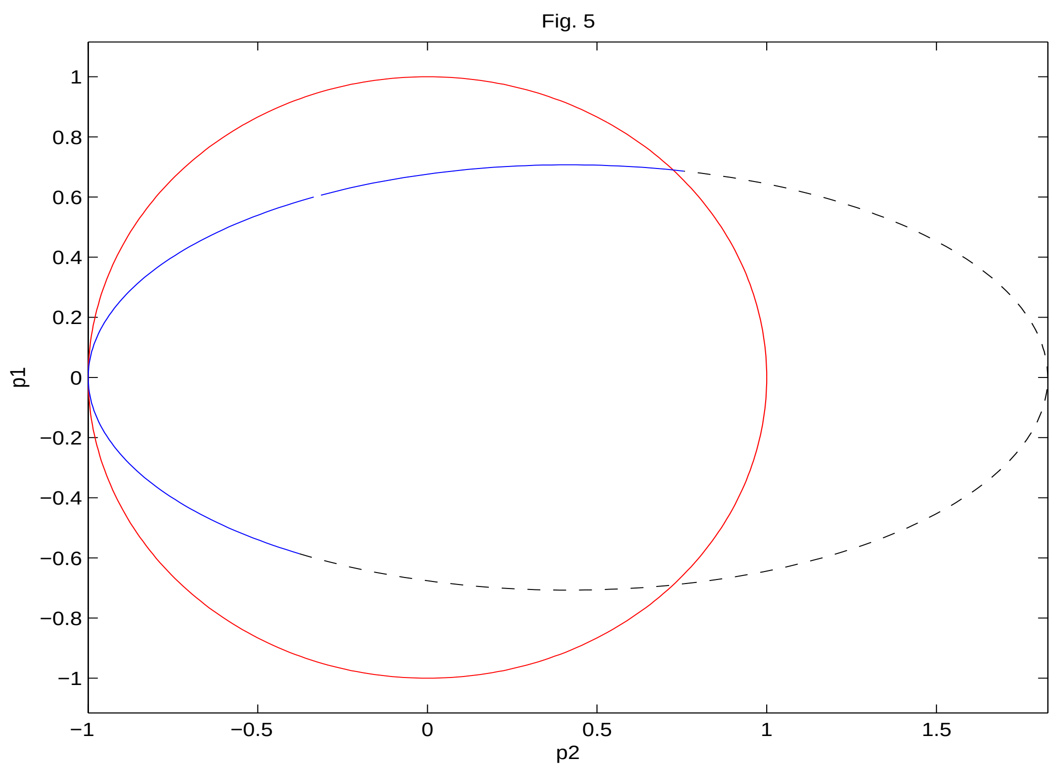

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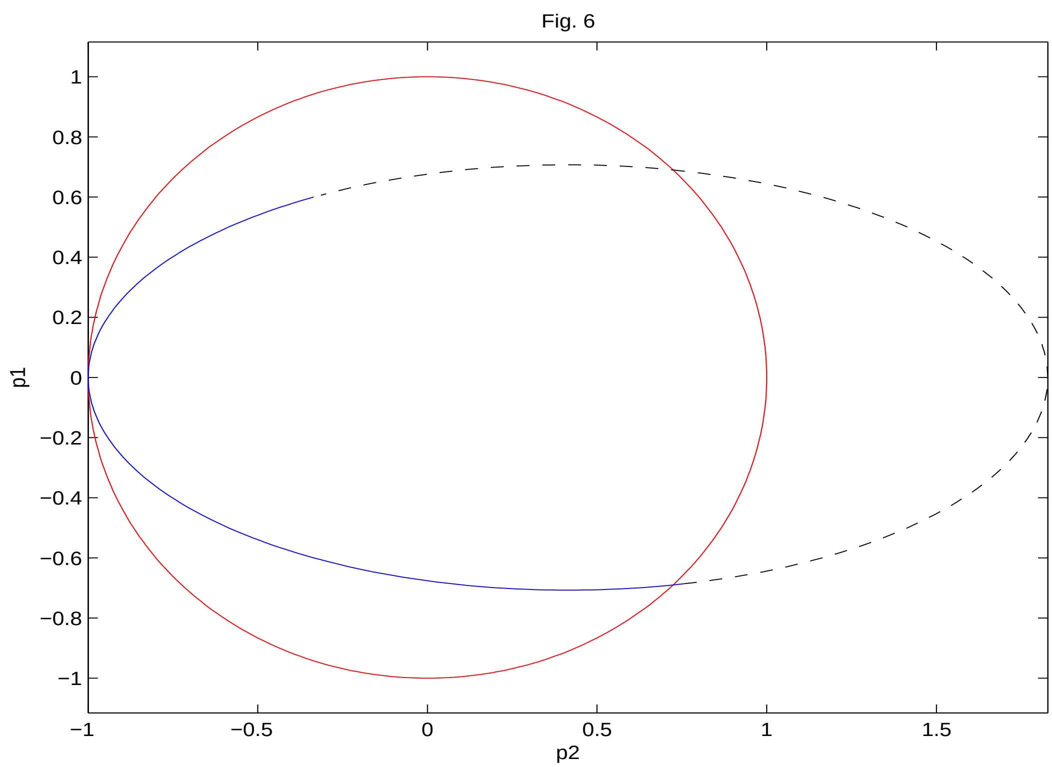

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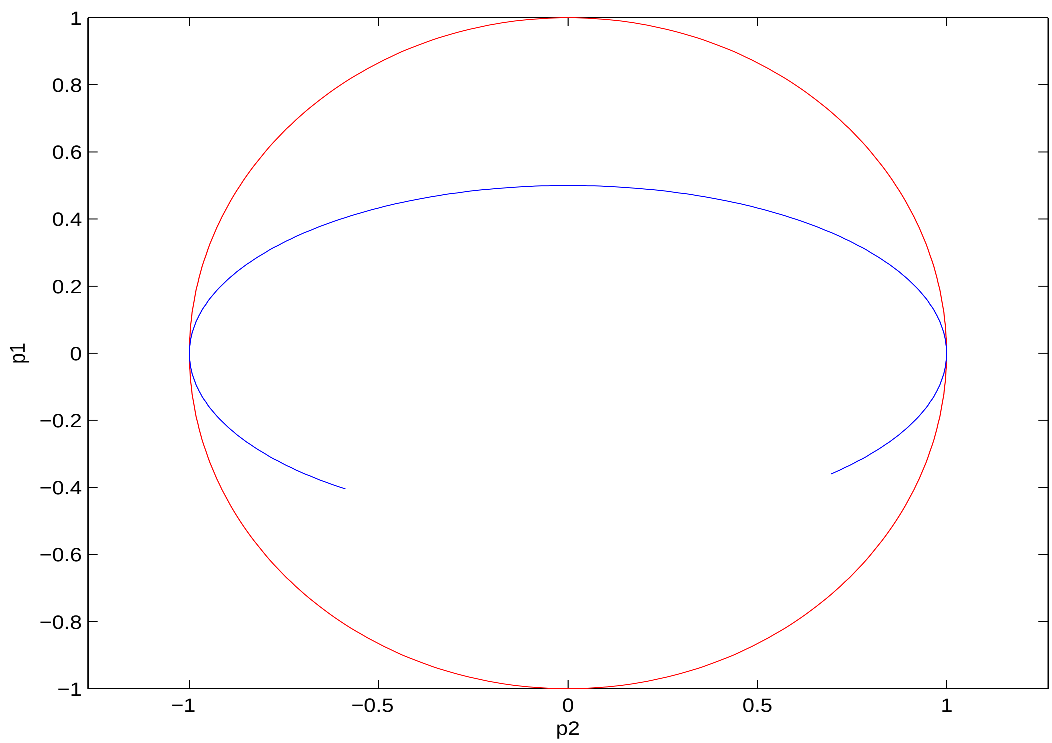

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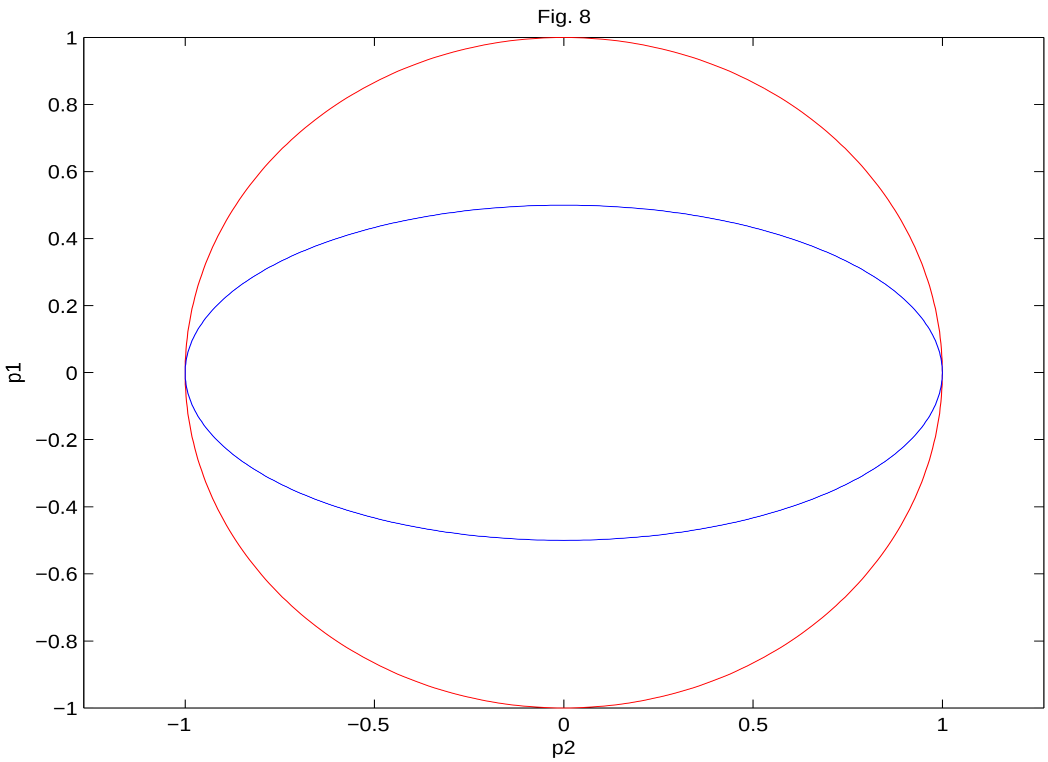

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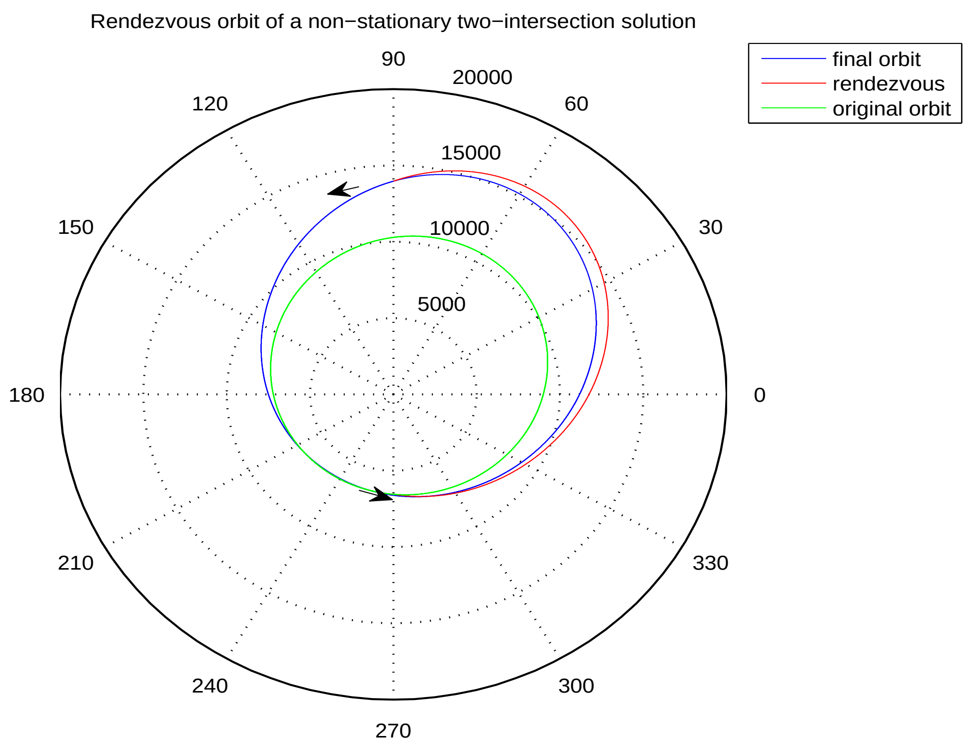
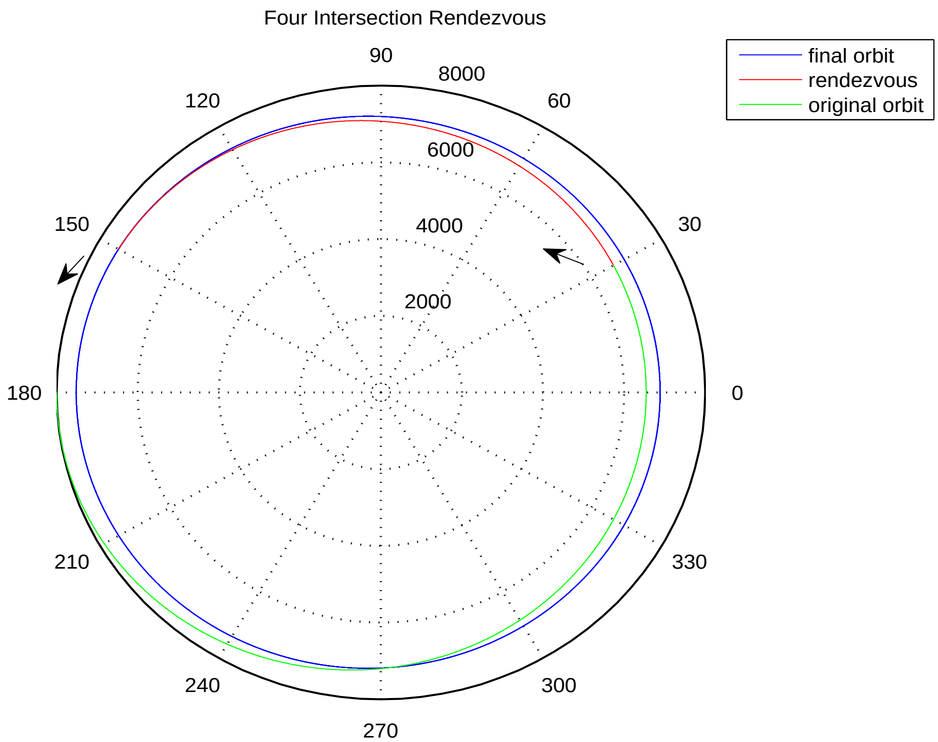

Rendezvous orbit of a non−stationary two−intersection solution
final orbit
rendezvous
original orbit
90
120
60
150
30
180
0
210
330
240
300
270
20000
15000
10000
5000

Figure 18:



Four Intersection Rendezvous
90
8000
60
120
6000
150
4000
30
2000
180
0
210
330
240
300
270
final orbit
rendezvous
original orbit

Figure 19:

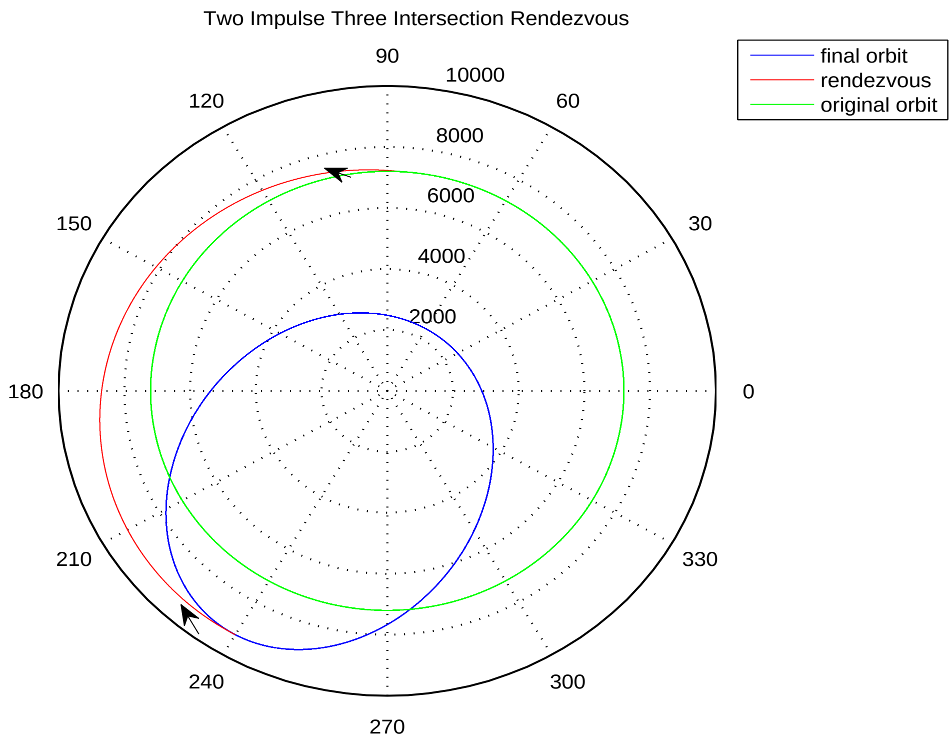

Two Impulse Three Intersection Rendezvous
final orbit
rendezvous
original orbit
90
60
30
0
330
300
270
240
210
180
150
120
10000
8000
6000
4000
2000

Figure 20:

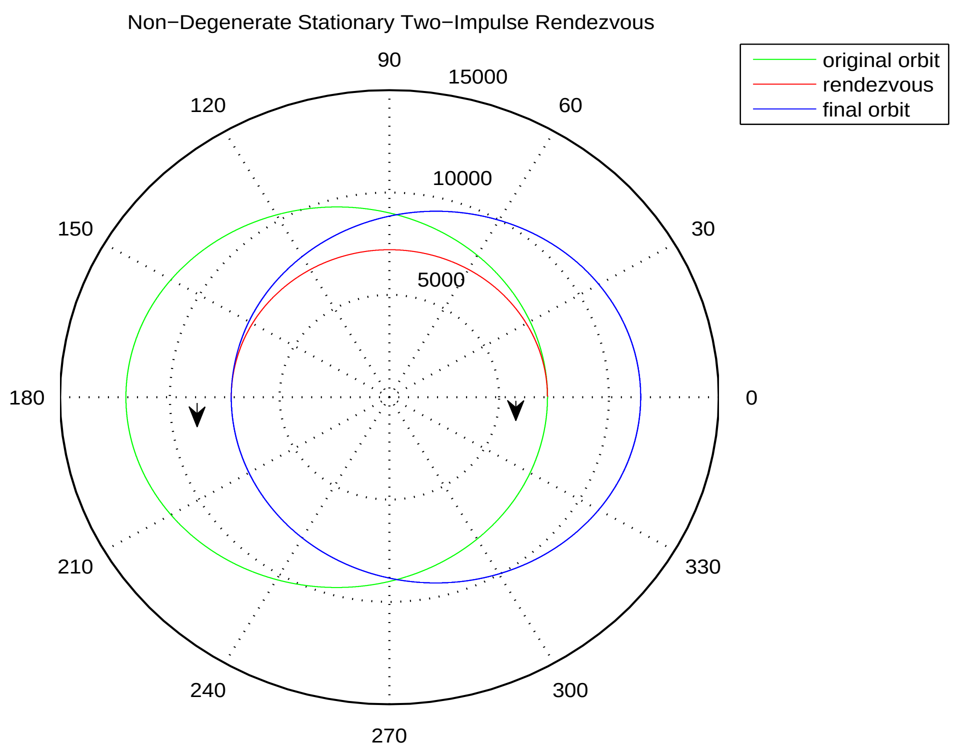

Non−Degenerate Stationary Two−Impulse Rendezvous
original orbit
rendezvous
final orbit
90
15000
60
120
10000
30
150
5000
180
0
210
330
240
300
270

Figure 21: